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The Nonlinear Gain and the Onset of Chaos in a Semiconductor Laser with Optical Feedback

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Abstract - The effect of the inclusion of nonlinear gain saturation in the governing equations of a semiconductor laser with optical feedback is numerically analyzed. The dependence of the route to chaos on the intensity of the nonlinear term and on the form of gain saturation, are considered. It is found that the inclusion of gain saturation causes significant changes in the behavior of the laser with respect to the route found when a linear optical gain is used [5]. We find that the value of the feedback rate at which the transition to coherence collapsed state occurs increases, as the intensity of the nonlinear term is augmented. In order to study the influence of the form of gain saturation, three different forms of optical gain, that result from spectral and kinetic hole burning or carrier heating, with an explicit intensity dependence are considered. The route to chaos is the same for the three forms considered but the parameters at which transition between attractors occur depend slightly on the form of gain saturation used. The calculated Lyapunov exponents, dimension and entropy confirm the fractal and chaotic nature of the attractors found.

1. INTRODUCTION

The complex laser dynamics induced by delayed feedback provides a rich variety of nonlinear phenomena. In a semiconductor laser with optical feedback intermittence [1], quasi periodicity [2], period-doubling [3] and coexistence of attractors [4] have been found experimentally and reported in numerical simulations. It is well known that the presence of a time delay in the feedback loop might cause the laser to switch to a state of significantly increased phase and dynamical complexity. This state has been called coherence collapsed state and has received a lot of attention from the experimental and the theoretical points of view. We recently showed that the laser dynamics in the coherence collapsed state is chaotic, calculating the Lyapunov spectrum and fractal dimension of the underlying attractor [5].

However, the laser behavior vary strongly when a nonlinear gain saturation is considered. The transient frequency dynamics of a gain-switched laser diode has been studied in [6], while a high order expression for the gain dependence on the carrier density and on the optical intensity is derived in [7] and is used to describe the semiconductor laser above and below threshold in the static and transient regimes. In [5] we studied the dynamics of the laser using the model of Land and Kobayashi [8] with the linear optical gain $G(N)=g(N-N_0)$. In this paper we present a detailed analysis of the effect of gain saturation on the chaotic dynamics of the laser. We find that the route to chaotic behavior of the laser with nonlinear gain is different than the one found in [5] where a linear gain was considered. Coexistence of two, and in some cases three, different attractors is reported. The value of the feedback rate at which the transition to coherence collapsed state occurs increases as the intensity of the nonlinear term in the optical gain is augmented. In fact, as the nonlinear term is increased, the attractors become more stable and the transition to the coherence collapsed state is delayed, i.e., increasing the intensity of the nonlinear term (decreasing the optical gain) seems to be equivalent to decreasing the feedback rate.

In order to analyze the influence of the form of gain saturation we have considered three different forms that have been previously used for studying transient and stationary regimes of semiconductor lasers with satisfactory results [9,10,2]. Although the inclusion of gain saturation has important effects on the dynamics of the laser, the dynamics is almost the same for the three forms of nonlinear gain considered and only the values of the feedback rate at which transition between attractors occur, depend slightly on the form used. This behavior is explained because for typical values of carrier density and optical intensity in the chaotic attractors, the three forms of optical gain take almost the same values.

The nature of the chaotic attractors found is studied using several characterization techniques associated with Poincaré sections, Lyapunov exponents, dimension and entropy. It is shown that using these techniques we can distinguish in the behavior of the laser different regimes of operation such as quasi periodicity with two incommensurate frequencies, quasi periodicity with three incommensurate frequencies, chaotic behavior and hyper-chaotic behavior.

The outline of this paper is as follows. In section 2 we describe the model equations and its main characteristics. In Section 3 we study the evolution of the laser when the feedback rate is increased, for different values of the intensity of the nonlinear term in the optical gain. Section 4 is devoted to the characterization of the chaotic dynamics of the laser, calculating the Lyapunov spectrum, dimension and entropy of the attractors found. The conclusion are presented in section 5.

2. THE MODEL EQUATIONS

The rate equations governing the behavior of a laser diode with weak to moderate feedback are the Lang and Kobayashi equations [8] for the amplitude $E_0(t)$ and the phase $\phi(t)$ of the electric field and the average carrier density N(t) in the active region. The equations are nonlinear and the field equation contains a time-delayed term that accounts for the field reflected from the external mirror and that renders the system infinite dimensional.

$$\frac{dE_o(t)}{dt} = \frac{1}{2}\left[G(N,E^2) - 1/\tau_p\right]E_o(t) + \frac{k}{\tau_{in}}E_o(t-\tau)\cos\left[\Delta(t)\right] \qquad (1)$$

$$\frac{d\phi(t)}{dt}=\frac{1}{2}\alpha\left[G(N,E^2)-1/\tau_p\right]-\frac{k}{\tau_{in}}\frac{E_o(t-\tau)}{E_o(t)}sin\left[\Delta(t)\right], \tag{2}$$

$$\frac{dN(t)}{dt}=J-\frac{N(t)}{\tau_s}-G(N,E^2)E_o(t)^2 \tag{3}$$

In these equations the field amplitude $E_0(t)$ is normalized such that $V_cE_0(t)^2$ is the total photon number in the laser wave-guide (where V_c is the volume of the active region). τ is the delay time (τ=L/2c where L is the length of the optical path and c is the velocity of light), τ_s is the carrier lifetime or population inversion lifetime, τ_p is the photon lifetime and τ_{in} is the round-trip time in the laser cavity. $G(N,E^2)$ is the optical gain and $\Delta(t)=\omega_0\tau+\phi(t)-\phi(t-\tau)$ is the phase delay (where ω_0 is the laser frequency without feedback at the threshold of the laser operation). k is the feedback rate, i.e., k^2 is the power reflected from the external cavity relative to the power reflected from the laser mirror. These equations do not include multiple reflection, and are therefore valid only for k<<1. α is the linewidth enhancement factor and J is the bias current.

Three different forms of nonlinear gain are used,

$$G_a(N,E)=\frac{g(N-N_o)}{\sqrt{1+2\varepsilon E^2}},\quad G_b(N,E)=\frac{g(N-N_o)}{1+\varepsilon E^2},\quad G_c(N,E)=g(N-N_o)(1-\varepsilon E^2),$$

with g being the modal gain coefficient, N_0 the carrier density at transparency and $\varepsilon=I_{sat}^{-1}$ is the inverse of the saturation intensity and will be taken as a parameter in order to increase the value of the nonlinear term. G_a was recently proposed by Agrawal [9] as providing the best fit to experimental measures of transient behavior while G_b and G_c are nonlinear gain saturations commonly used [10,2]. The three forms of gain saturation become equivalent by expanding to first order in εE^2.

3. DYNAMICS OF THE MODEL

The behavior of the model when the gain saturation increases is studied in detail. The parameter ε was varied as $\varepsilon=\varepsilon_p\,\varepsilon_{max}$ with ε_p in [0,1] and ε_{max} = 7,5x10^{-23} m^3. We numerically integrated Eqs. (1)-(3) using G_a form of optical gain and a six-order Runge-Kutta integration routine with a time increment Δt=0.01ns. The values of the parameters used are the same as [5]: α=6, g=1.1x10^{-12} m^3/s, N_o=1.1x10^{24} m^3, τ_s=2 ns, τ_p=2 ps, τ_{in}=6.7 ps, τ=2 ns, J=2.0J_{th} where J_{th} is the threshold current. We use the feedback rate k as the control parameter, in order to study the laser's route to chaotic behavior.

The stationary solutions of eqs. (1)-(3) are the external cavity modes (ECM) of the laser. If we neglect the small contribution from nonlinear gain [11], the stationary angular frequencies ω_s are the solutions of the equation

$$\omega=\omega_o-\frac{k}{\tau_{in}}\sqrt{1+\alpha^2}\,sin(\omega\tau+arctan\,\alpha) \tag{4}$$

while the stationary carrier density and field amplitude corresponding to a given value of ω_s are

$$N_s = N_{th} - \frac{2k}{g\tau_{in}} \cos\omega_s\tau \;, \tag{5}$$

$$E_s^2 = \frac{1}{g(N - N_o)}(J - \frac{N_s}{\tau_s}). \tag{6}$$

where N_{th} is the threshold carrier density. The evolution of one ECM is studied for an increasing feedback level. The computer program was initiated from the ECM with minimum carrier density, and was first started for the laser diode without feedback. The value of the electric field in one round-trip interval is stored in memory in order to use it in next round-trip interval. We analyzed the trajectory after a large number of round trips in order to eliminate transient effects.

Without feedback, the system shows the characteristic damped relaxation oscillations to the ECM of the laser. With delayed feedback, different motions might occur since the laser becomes a system with infinite dimension. Increasing the feedback level k, the ECM becomes unstable and a limit cycle appears. If we continue to increase k, the laser follows a quasiperiodic route and a two-torus appears. This attractor will be called attractor A. The evolution of this attractor when k is increased, i.e., the system's route to chaos, depends on the intensity of the nonlinear gain saturation considered ε.

Some typical Poincaré sections [12,13] that illustrate the evolution of the laser are shown in Figs. 1 and 2. These Poincaré sections were obtained plotting the normalized photon number $N(t)/N_{th}-1$ and the phase delay $\phi(t)-\phi(t-\tau)$ at the intersection points of the trajectory with the plane $I(t)/I_s=E(t)^2/E_{sol}{}^2=1$.

Fig. 1 illustrates the evolution of attractor A for $\varepsilon_p=0.2$. For low values of k, the attractor is a two torus (Fig. 1 a). For higher values of k, a third incommensurate frequency appears and attractor A becomes a three-torus (Fig. 1 b). It is interesting to notice that, when using a linear gain in [5] was found that the two torus A undergoes several period-doubling bifurcations before becoming chaotic. Therefore, the inclusion of a small nonlinear term in the optical gain changes the period-doubling route into a quasi-periodic route and a three torus appears.

If we continue to increase the feedback rate, above a certain critical point attractor A becomes unstable and a new attractor, which will be called attractor B, appears. Notice that while the Poincaré section of attractor A has phase delay $\phi(t)-\phi(t-\tau) < 0$ (Figs. 1 a, b), the Poincaré section of attractor B has $\phi(t)-\phi(t-\tau) >0$ (Figs. 1 c, d). In [5] using a linear gain and the same parameters as this paper, no coexistence of attractors was found.

Attractor B is a two torus that undergoes a period-doubling bifurcation (Fig. 1 d) before the chaotic regime appears (Fig. 1e). For higher values of the feedback (Fig. 1 f), the trajectory "jumps" between the vicinity of attractors A and B. This regime is identify with the coherence collapsed state and is strongly reminiscent of the intermittence phenomena.

For larger values of ε a new torus appears (attractor C) and laser behavior becomes more complicated. Fig. 2 illustrates this behavior for $\varepsilon_p=0.5$. We can identify different types of regimes where the laser operates, that depend on the type of stable attractor. For low values of k, the stable attractor is attractor A (regime I, Fig. 2 a, b). When this attractor becomes unstable, the laser switches to attractor B, which is the stable attractor in this regime (II, Fig. 2 c). Attractor B period-doubles before becoming chaotic. This chaotic regime has Poincaré section with values $\phi(t)-\phi(t-\tau) >0$ (Fig. 2 d) and is desestabilized because of the presence of a new

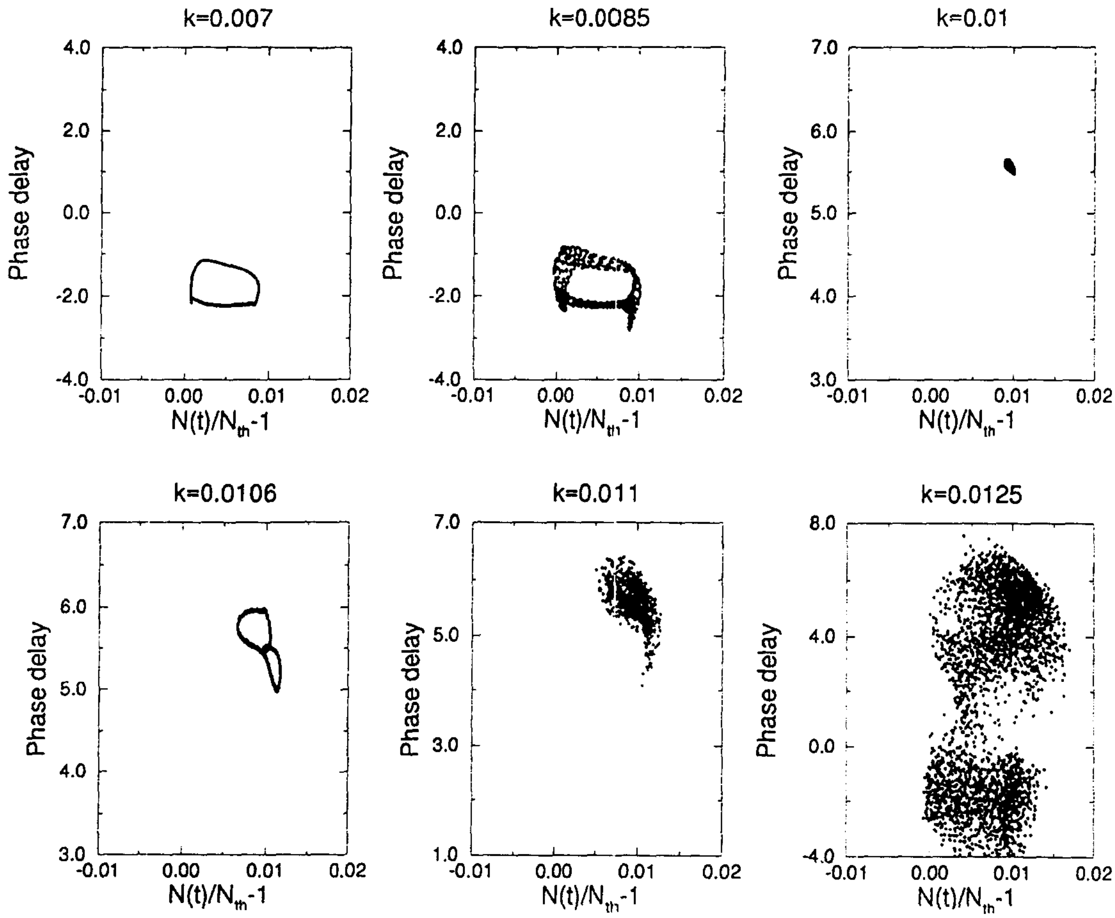

Fig. 1 Poincaré sections for e_p=0.2. (a) two-torus A (the Poincaré section has values $\phi(t)$-$\phi(t-\tau)$<0) ; (b) three-torus A; (c) two-torus B (the Poincaré section has $\phi(t)$-$\phi(t-\tau)$>0) ; (d) torus B that period doubled; (e) chaotic attractor B and (f) coherence collapsed state (the trajectory jumps between attractors A and B)

stable attractor. This new attractor (C, Fig. 2 e) is a two dimensional torus that has Poincaré section with values $\phi(t)$-$\phi(t-\tau)$ >0 but that are slightly above of those of attractor B, i.e., attractor C is in the "middle" of attractors A and B. Regime III corresponds to the region when the laser operates close to attractor C. Finally, when attractor C becomes unstable, the laser switches between the three unstable attractors. This is regime IV shown in Fig. 2 f, that corresponds to the coherence collapsed state of the laser. The Poincaré section has points with $\phi(t)$-$\phi(t-\tau)$ >0 (close to the unstable torus B or C) and points with $\phi(t)$-$\phi(t-\tau)$ <0 (close to the unstable torus A).

Let us now study the effect of increasing the nonlinear term in the optical gain for a fixed value of k. Fig. 3 shows the phase portrait and the Poincaré section of attractor A, for k=0.007 and different values of ε_p.

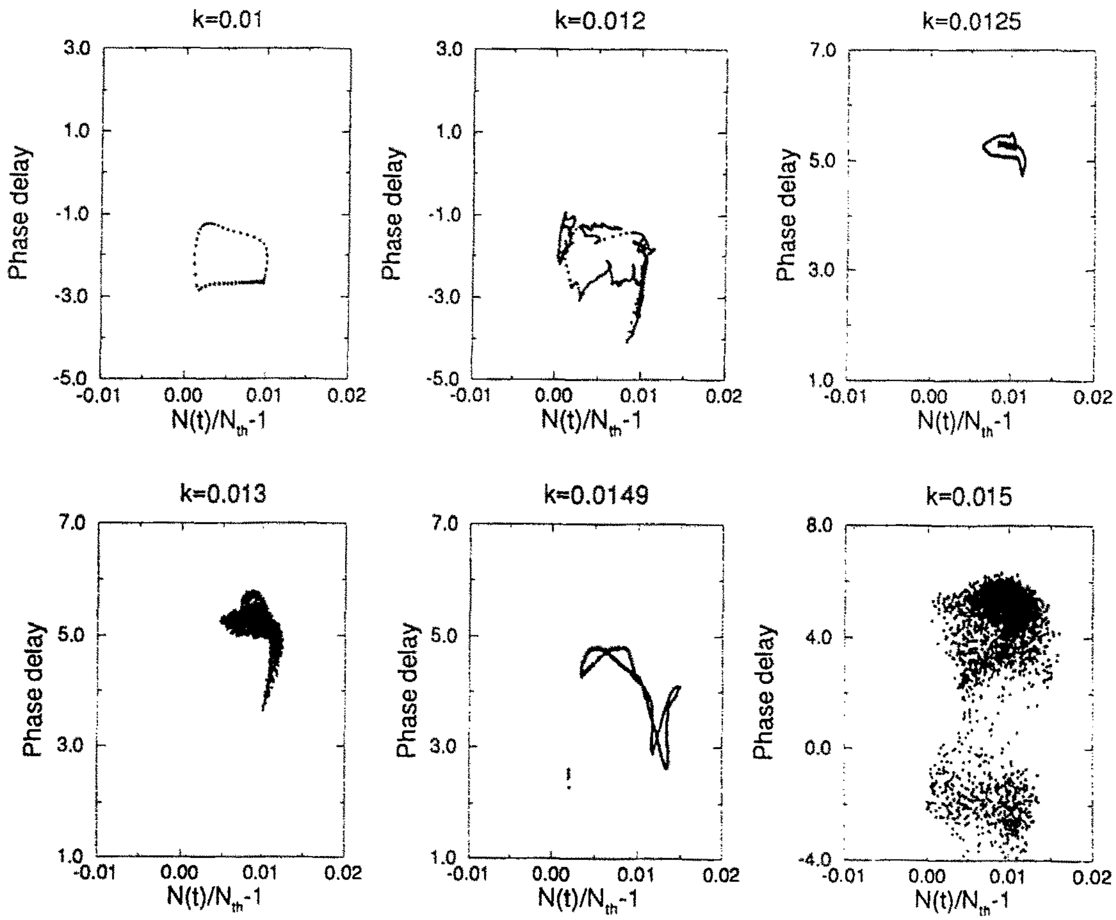

Fig. 2 Poincaré sections for ε_p=0.5. (a) two-torus A; (b) two-torus A deformed; (c) torus B that period doubled; (d) chaotic attractor B; (e) two-torus C and (f) coherence collapsed state

In the linear case (ε_p =0) the attractor is chaotic (regime IV). As we increase the value of ε_p the attractor becomes more "ordenated" and is a two-torus for ε_p=0.5 and a limit cycle for ε_p=0.8. Thus, increasing the intensity of the nonlinear term (i.e., decreasing the optical gain) for a fixed value of k seems to be equivalent to decreasing the feedback rate for a fixed value of ε_p.

In order to study the influence of the form of optical gain in the dynamics of the system, we have integrated Eqs. (1)-(3) using G_b and G_c forms. We have found almost the same behavior as the one described previously. The values of the feedback rate at which transition between attractors occur depend slightly on the gain saturation form used, but the shape of the attractors and the route to chaotic behavior is the same that the one described previously for the form G_a. This is explained because for low values of ε as the ones used in this paper and the values of carrier density and optical intensity in the attractors (typically $|N(t)/N_{th}-1|<0.02$ and $0<E(t)^2/E_{sol}^2<2$, see Fig. 3), the three forms of optical gain take approximately the same values.

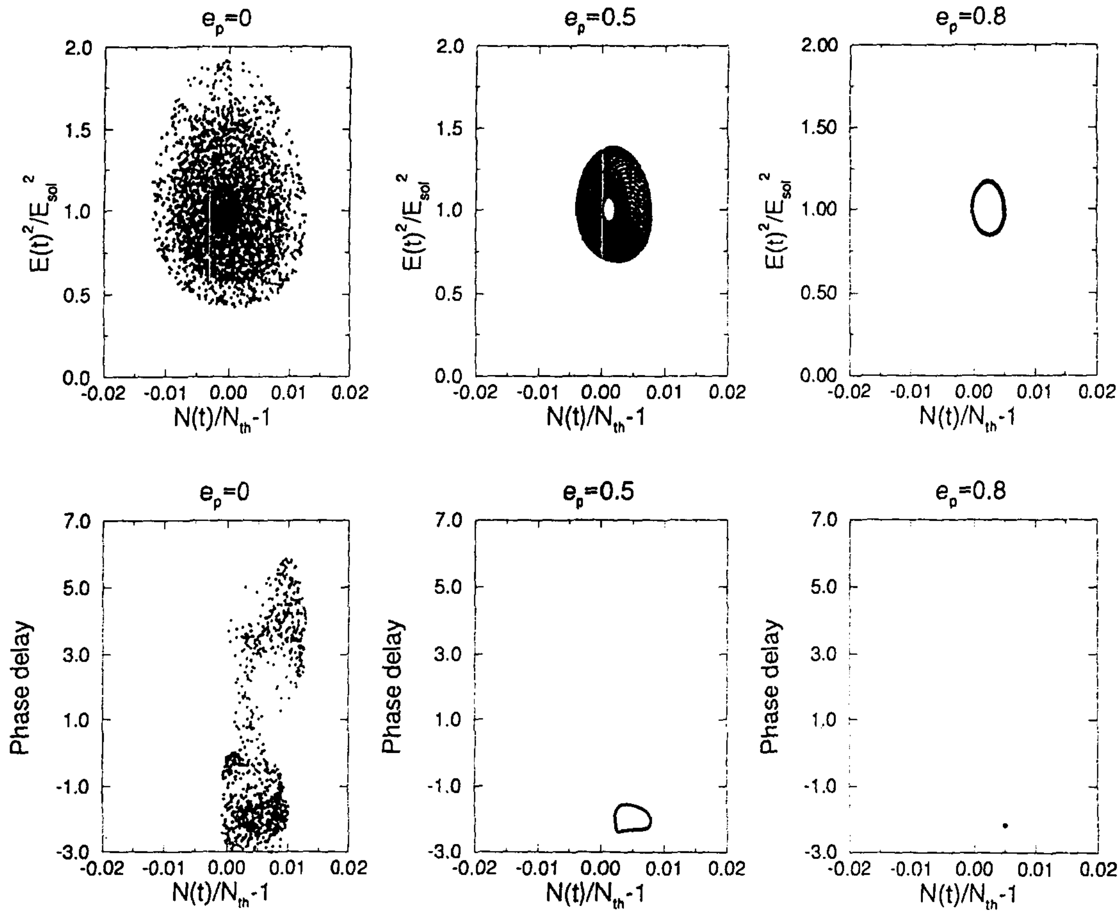

Fig. 3 (a), (b), (c) Phase plots of the trajectory; (d), (e), (f) Poincaré sections for k = 0.007 and ε_p = 0; 0.5 and 0.8 respectivelly. For ε_p=0 the attractor is chaotic while for ε_p=0.5 is a two-torus and for ε_p=0.8 a limit cycle.

4. CHARACTERIZATION OF THE MODEL'S ATTRACTORS

In order to gain more information about the stability and topological properties of the attractors found in the previous section, we have calculated their Lyapunov exponents, dimension and entropy.

The Lyapunov exponents of an attractor show qualitatively the sensitive dependence on initial conditions by measuring the average separation speed of nearby trajectories on the attractor. Since a semiconductor laser with delayed feedback is a system with infinite dimensions, we must study the evolution of infinitesimal perturbations that are vectors with three components, two of which (the field and the amplitude) are functions of time over the entire delay τ. The technique we used is as follows [5, 14, 15]: for each exponent λ^i to be computed, we arbitrarily selected an initial perturbation $\mathbf{dx^i(0)}$ = ($dE^i(0)$, $d\phi^i(0)$, $dN^i(0)$). We monitored the evolution of the trajectory $\mathbf{x(t)}$ = ($E(t)$, $\phi(t)$, $N(t)$) and of the perturbations $\mathbf{dx^i(t)}$ integrating simultaneously the nonlinear evolution equations (1)-(3) as well as the linealized equations, for a round-trip time τ. Then we normalized the first perturbation $\mathbf{dx^1}$ to have length one and using a Gram-Schmidt algorithm, we orthonormalized the second vector relative to the first, the third relative to the first and second, and so on. We repeated this procedure for L round trip times and computed

$$\lambda^i = \frac{1}{L\tau} \sum_{k=1}^{L} Log \left\| dx^i(k) \right\|, \tag{7}$$

where the Euclidean metric was chosen to define distance in the phase space

$$\left\| dx^i(k) \right\|^2 = \frac{1}{n}\sum_{j=1}^{n} \left(\frac{dE_j^i(k)}{E_o}\right)^2 + \frac{1}{n}\sum_{j=1}^{n} (d\phi_j^i(k))^2 + \left(\frac{dN^i(k)}{N_o}\right)^2. \tag{8}$$

Here dE_j^i and $d\phi_j^i$ with j=1...n are the electric field and phase components of the i-perturbation over the entire delay interval τ.

Since Eqs. (1)-(3) are invariant under a global translation in the phase ($\phi_j \rightarrow \phi_j + a$), two points in the attractor that differ in a constant value of ϕ will not merge nor separate. Thus, the system has a "spurious" zero Lyapunov exponent in the spectrum that corresponds to a fixed zero in the system determinant D(s) [11]. This exponent does not influence the stability of the attractors and will be ignored in the rest of our calculations.

The fractal dimension of an attractor is related to the number of collective degrees of freedom exited in the system. From the spectrum of Lyapunov exponents we can calculate the Lyapunov dimension using the definition of Kaplan and Yorke [16]

$$D_L = j + \frac{\sum_{i=1}^{j} \lambda^i}{\left|\lambda^{j+1}\right|} \tag{9}$$

where $\lambda^1 > \lambda^2 > ... > \lambda^n$ and j is the largest integer for which $\lambda^1 + \lambda^2 + ... + \lambda^j \geq 0$. For typical attractors it has been conjectured [17] that the Lyapunov dimension is equal to the well known information dimension.

Finally, the metric entropy, that provides information on a typical predictability time of the motion, can also be computed from the spectrum of Lyapunov exponents. The relation between the metric entropy h_μ and the Lyapunov exponents [18] of an attractor is

$$h_\mu = \sum_{i=1}^{j} \lambda_+^i \tag{10}$$

where λ^i_+ are the positive Lyapunov exponents of the attractor.

The results obtained are summarized in Tables I and II for the attractors shown in Figs. 1 and 2 respectively.

For ε_p=0.2 and k=0.007, the attractor has two exponents equal to zero, Lyapunov dimension equal to two and zero metric entropy, i.e., is a two dimensional torus (see Fig. 1a). For k=0.0085 the attractor is a three torus that has three exponents equal to zero, dimension equal to three and zero entropy. For k=0.01 and k=0.0106 the attractor is the two dimensional torus B, while for k=0.011 and k=0.0125 the attractors are chaotic. The Lyapunov exponents and metric entropy calculated confirm this fact, since the attractors have at least one positive exponent and positive entropy.

For ε_p=0.5 the results obtained are summarized in Table II. For k=0.01 and 0.012 the attractor is two torus A, while for k=0.0125 the attractor is two torus B and for k=0.013 the chaotic attractor B. For k=0.0149 the attractor is the two dimensional torus C, and finally, for k=0.015 the attractor is chaotic.

It is important to notice that the regime IV, that was identified with the collapsed coherence state, is a hyper chaotic state that has two or more positive Lyapunov exponents. The dynamical complexity of this state is evidenced by the fact that is has larger fractal dimension and entropy than the chaotic states shown in Fig. 1e (ε_p=0.2, k=0.011) or Fig. 2 d (ε_p=0.5, k=0.013).

TABLE I: Lyapunov exponents of the attractors shown in Fig. 1 (ε_p=0.2)

k		Lyapunov Exponents								D_L	h_μ
0.007	0	0	-0.12	-0.23	-0.23	-0.47	-0.47	-0.54	-0.67	2	0
0.0085	0	0	0	-0.05	-0.27	-0.28	-0.53	-0.56	-0.56	3	0
0.01	0	0	-0.02	-0.07	-0.12	-0.30	-0.30	-0.43	-0.46	2	0
0.0106	0	0	-0.02	-0.02	-0.17	-0.3	-0.3	-0.48	-0.48	2	0
0.011	0.13	0	0	-0.04	-0.09	-0.26	-0.30	-0.39	-0.45	5	0.13
0.0125	0.34	0.12	0.02	0	-0.05	-0.12	-0.19	-0.24	-030	7.5	0.48

TABLE II: Lyapunov exponents of the attractors shown in Fig. 2 (ε_p=0.5)

k		Lyapunov Exponents								D_l	h_μ
0.0100	0	0	-0.12	-0.20	-0.20	-0.29	-0.33	-0.40	-0.46	2	0
0.0120	0	0	-0.07	-0.08	-0.15	-0.27	-0.28	-0.34	-0.39	2	0
0.0125	0	0	-0.09	-0.18	-0.19	-0.29	-0.29	-0.36	-0.36	2	0
0.0130	0.12	0	0	-0.15	-0.19	-0.23	-0.26	-0.33	-0.35	4.77	0.12
0.0149	0	0	-0.01	-0.01	-0.06	-0.15	-0.23	-0.25	-0.25	2	0
0.0150	0.10	0.03	0	-0.01	-0.06	-0.13	-0.17	-0.22	-0.25	5.46	0.13

5. CONCLUSIONS

We have analyzed in detail the effects of the inclusion of nonlinear gain saturation in the governing equations of a semiconductor laser with optical feedback. We found that gain saturation varies strongly the behavior of the laser with respect of the behavior found in [5] where a linear optical gain was used.

We found that, as could be expected, when the intensity of the nonlinear term is increased, for a fixed value of the feedback rate, the attractor is stabilized. In fact, the chaotic attractor turns into a two dimensional torus and this torus turns into a limit cycle when the parameter ε is increased and the feedback rate k remains constant (see Fig. 3). This effect simulates the decreasing of the feedback rate for a fixed value of the nonlinear term.

In addition, on the contrary with the linear case, when a nonlinear gain saturation is used, coexistence of attractors is found for a large range of the parameters. Moreover, for particular values of these parameters, intermittence is found and the trajectory jumps randomly between several unstable attractors .

Finally, the characterization of the different attractors found was done, calculating their Lyapunov exponents, fractal dimension and entropy. With this characterization we can clearly discriminate between the different regimes in which the laser operates, such as two-frequency regime, three-frequency regime, chaotic and hyper-chaotic regimes.

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