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Numerical simulations of the effect of noise on a delayed pitchfork bifurcation

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Abstract

We investigate the effect of noise on a delayed pitchfork bifurcation by numerical simulations of the Langevin rate equation. When the delay is calculated averaging over a large number of trajectories that differ only on the noise realization, our results are in very good agreement with the analytical results reported in the literature (H. Zeghlache et al., Phys. Rev. A 40 (1989) 286). However, there are regions of the parameters and initial conditions for which the delay of a single trajectory is strongly influenced by noise, even for very low noise levels. Our numerical results show that in these regions the noise level at the moment when the trajectory passes the static bifurcation point might be what mainly determines the posterior evolution and delay.

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Keywords: Bifurcations; Noise

The delay introduced by the linear sweep of a parameter through a critical point has been studied by several authors. The experimental situation has been realized in lasers, where at threshold the zero intensity solution loses stability and the nontrivial solution becomes stable. The vicinity of this critical point is characterized by critical slowing down, and the dynamics is dominated by a characteristic decay time which is not related to the atomic and cavity decay times but it is of geometrical origin (and diverges at the critical point) [1]. The delay in the laser turn-on was first studied (by varying in time the cavity losses) in an argon ion laser [2]. More recently, delayed bifurcations were studied in dye lasers [3], CO_2 lasers [4,5], and semiconductor lasers [6,7].

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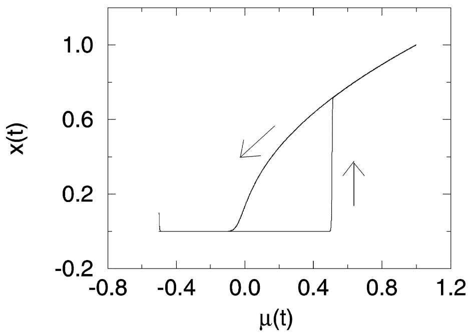

Fig. 1. Delayed bifurcation and hysteresis phenomenon when the parameter μ is increased and decreased. We plot the numerical solution $x(t)$ as a function of $\mu(t)x_0 = 0.1$, $\mu_0 = -0.5$, $v = 0.001$, and $D = 0$.

The influence of noise on a delayed bifurcation was theoretically investigated by Mandel and coworkers, analyzing the Langevin equation by the method of moments (see e.g. Refs. [8,9]), and by San Miguel and coworkers, analyzing the Fokker–Plank equation (see the review paper [10]). In general, noise reduces the delay even to the extent that the bifurcation point can occur before the static bifurcation point.

In this paper, we investigate the equation

$$dx/dt = x(t)[\mu(t) - x(t)^2] + \psi(t) \tag{1}$$

when $\psi(t)$ is a small noise source, and the parameter μ is increased linearly in time from a value $\mu_0 < 0$ to a value $\mu_f > 0 : \mu(t) = \mu_0 + vt$.

We perform simulations considering white noise, $\langle\psi(t) \cdot \psi(t')\rangle = 2D\delta(t - t')$, and colored noise, $\langle\psi(t)\psi(t')\rangle = (D/\tau)\exp(-|t - t'|/\tau)$. We use the algorithm described in Ref. [10] for the numerical generation of trajectories (Eq. (2.75) of Ref. [10] for the integration of a stochastic differential equation with additive white noise, Eq. (2.102) to generate colored noise, and Eq. (2.107) for the integration of a stochastic differential equation with additive colored noise). We compare our numerical results with the analytical results of Ref. [9].

In the static deterministic case (when the parameter μ is constant in time and $\psi(t) = 0$), the stable solutions of Eq. (1) are $x = 0$ if $\mu < 0$, and $x = \pm\sqrt{\mu}$ if $\mu > 0$. When Eq. (1) is integrated with initial conditions $x_0 < 0$, $\mu_0 < 0$, the linear sweep of the parameter μ induces a delay in the bifurcation that is shown in Fig. 1. The initial condition x_0 relaxes to $x = 0$ when $\mu(t) < 0$ but when μ becomes positive the numerical solution does not evolve immediately towards one of the new solutions ($x = \pm\sqrt{\mu}$) but there is a delay in which, for $0 < \mu < \mu^*$ the numerical solution continues evolving along the branch $x = 0$. For a certain value of the parameter ($\mu = \mu^* > 0$), the numerical solution leaves the vicinity of the trivial solution and rapidly evolves towards one of the nontrivial solutions ($+\sqrt{\mu}$ or $-\sqrt{\mu}$). The effect of the sweep of the parameter is therefore to induce a stability in the zero solution, which leads to a shift (or a delay) of the critical point from its original (static) position. If the parameter μ is now linearly decreased from $\mu_f > 0$ to $\mu_0 < 0$, a hysteresis phenomenon occurs, since the numerical solution follows the solution $x = +\sqrt{\mu}$ or $x = -\sqrt{\mu}$ for $\mu > 0$, and $x = 0$ for $\mu < 0$.

The delay in the bifurcation can be defined as $\mu^* = \mu(t^*)$ with the delay time $t^* > 0$ such that $\langle x(t^*)^2 \rangle \geqslant x_{\text{th}}^2$, where x_{th} is a prescribed value (t^* is the first passage time: the escape time from the unstable $x = 0$ state). A disadvantage of this definition if that the minimum delay that can be measured is $\mu^*_{\text{min}} = x^2_{\text{th}}$ (shortly after the delayed bifurcation, $x(t) = \pm\sqrt{\mu(t)}$, and if $\mu^* < \mu^*_{\text{min}}$, then $x(t)^2$ will be larger than x^2_{th} only when $\mu(t)$ becomes larger than x^2_{th}).

In Ref. [9] the cubic term of Eq. (1) was not included, and analytical expressions for the time evolution of $\langle x(t)^2 \rangle$, and for the delay, μ^*, were obtained. Choosing $x_{\text{th}} = x_{\text{o}}$, leads to $\mu^* = -\mu_0$ in the deterministic case. Therefore, when no noise is added to Eq. (1), the delay increases linearly with the initial condition of the parameter μ_0, and is independent of the initial condition of the variable, x_0, and of the swept rate, v.

In the presence of white noise, three regions can be distinguished, depending on the value of μ_0. If μ_0 is not too large the delay μ^* increases linearly with μ_0. For larger values of $|\mu_0|$ the effect of noise increases, and the delay is reduced compared to the deterministic delay. The effect of noise saturates, and for large enough values of $|\mu_0|$ the delay is independent of μ_0. In this case, μ^* depends on the noise level, the swept rate, and the initial condition x_0 through the factor $b = (D/x_0^2)\sqrt{\pi/v}$ [9].

To check these results we performed extensive numerical simulations. In the simulations and in the presence of noise, care must be taken on how to calculate the delay from individual trajectories, and how to chose the value of x_{th}. Following Ref. [9], we choose $x_{\text{th}} = x_0$. If x_0 is too small, then the condition $x(t)^2 \geqslant x_0^2$ might be satisfied because of the noise and not because a dynamic bifurcation occurred. On the other hand, if x_0 is too large, short delays will not be detected (since $\mu^*_{\text{min}} = x_0^2$). In addition, if the noise is strong enough $x(t)^2$ might initially be larger than x_0^2. To overcome this difficulty we defined t^* for an individual trajectory as $x(t^*)^2 \geqslant x_{\text{o}}^2$ and $x(t^*)\dot{x}(t^*, \psi = 0) > 0$.

Our results are shown in Fig. 2. In order to compare with the results of Ref. [9], we plot $\mu^*/\sqrt{v}$ vs. $\mu_0/\sqrt{v}$ for different noise levels (see Fig. 1 of Ref. [9]). We calculated the delay averaging over 60 trajectories which differ only on the noise realization. Clearly, our results are in good agreement with those of Ref. [9]. For low values of μ_0 the delay increases linearly with μ_0, while for larger values of $|\mu_0|$, there is saturation. The length of the 'deterministic region', where noise has almost no influence on the dynamics, is larger for lower noise intensities. Also, in agreement with the results of Ref. [9], if D, x_0, and v are changed such that the value of b remains unchanged, the delay remains nearly unchanged.

However, the delay of a single trajectory (shown in Fig. 3) presents large fluctuations when μ_0 is chosen in the saturation region, and almost no fluctuations for lower values of $|\mu_0|$ (for which the delay increases linearly with $|\mu_0|$).

The previous results can be understood by considering that for a single trajectory, the value of $x(t)$ when $\mu(t) = 0$ is what mainly determines the delay (we call this value $x_{\mu=0}$). If $x_{\mu=0}$ is above the noise level (either because μ_0 is small, or x_0 is large,

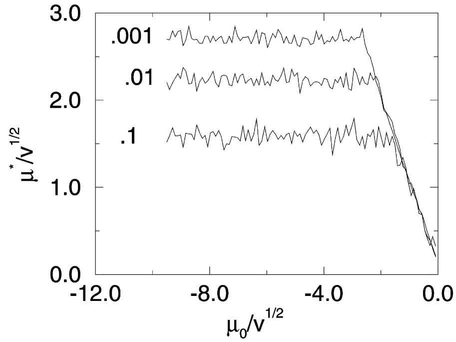

Fig. 2. Delay as a function of the initial condition of the parameter, μ_0, for $v = 0.001$, $x_0 = 0.05$, and three different noise levels: $D = 4.5 \times 10^{-8}$ ($b = 0.001$), $D = 4.5 \times 10^{-7}$ ($b = 0.01$), and $D = 4.5 \times 10^{-6}$ ($b = 0.1$).

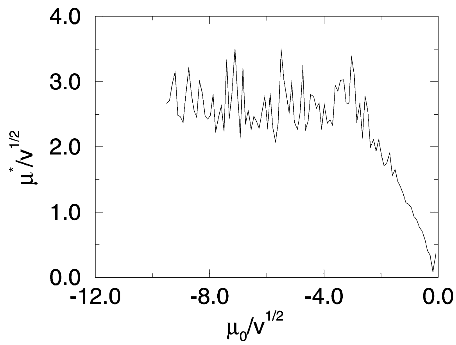

Fig. 3. Same as Fig. 2, but the delay is calculated from a single trajectory. $D = 4.5 \times 10^{-8}$.

or D is small), then the delay will not be strongly influenced by noise. However, if $x_{\mu=0}$ is smaller than the amplitude of the noise, then the noise level at the time when $\mu(t) = 0$ is what becomes amplified for $\mu > 0$ and is what mainly determines the posterior evolution and delay. Fig. 4 plots the delay as a function of the value of $x(t)$ when $\mu(t) = 0$. The plot is done considering a large number of trajectories which have different initial conditions, μ_0 and x_0. Clearly, if the value of $x_{\mu=0}$ is large enough the delay is a well-defined function of $x_{\mu=0}$, while if $x_{\mu=0}$ is small, there is almost no correlation between the value of $x_{\mu=0}$ and the value of μ^*.

When Eq. (1) was integrated with colored noise, with low noise intensities no difference was found between the effect of colored noise and the effect of white noise. However, for large noise intensities the correlation of the noise might lead to large deviations of the trajectory from the trivial solution, which make the concept of delayed bifurcation almost meaningless.

In conclusion, our numerical simulations show that the analytic results of Ref. [9] accurately estimate the delay of a single trajectory when the parameters and the initial conditions are chosen in the 'deterministic' region, and estimate an 'average' delay

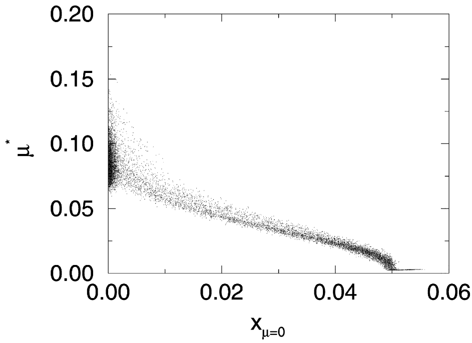

Fig. 4. Delay as a function of the value of $x(t)$ when $\mu(t)=0$. A large number of trajectories with different x_0, μ_0 are considered. The delay of a trajectory is defined as $x(t^*)^2 \geqslant x_{\mathrm{th}}^2$ with $x_{\mathrm{th}}=0.05$. Therefore, $\mu^*_{\mathrm{min}}=0.0025, v=0.001$ and $D=4.5\times 10^{-8}$.

when the parameters and the initial conditions are chosen in the region where noise is what mainly influences the delay.

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References

[1] P. Mandel, T. Erneux, Opt. Commun. 44 (1982) 55.
[2] W. Scharpf, M. Squicciarini, D. Bromley, C. Green, J.R. Tredicce, L.M. Narducci, Opt. Commun. 63 (1987) 344.
[3] G. Vemuri, Opt. Commun. 85 (1991) 36.
[4] F.T. Arecchi, W. Gadomski, R. Meucci, J.A. Roversi, Opt. Commun. 70 (1989) 155.
[5] D. Bromley, E.J. D'Angelo, H. Grassi, C. Mathis, J.R. Tredicce, S. Balle, Opt. Commun. 99 (1993) 65.
[6] S. Balle, M. San Miguel, N.B. Abraham, J.R. Tredicce, R. Alvarez, E.J. D'Angelo, A. Gambhir, K. Scott Thornburg, R. Roy, Phys. Rev. Lett. 72 (1994) 3510.
[7] S. Choi, E. White, D.M. Wood, T. Dodson, K.V. Vasavada, G. Vemuri, Opt. Commun. 160 (1999) 261.
[8] P. Mandel, Theoretical Problems in Cavity Nonlinear Optics, Cambridge University Press, Cambridge, 1997.
[9] H. Zeghlache, P. Mandel, C. Van der Broeck, Phys. Rev. A 40 (1989) 286.
[10] M. San Miguel, R. Toral, Stochastic effects in physical systems, in: E. Tirapegui (Ed.), Instabilities and Nonequilibrium Structures VI, Kluwer Academic, London, 1998.