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Numerical investigation of noise-induced resonance in a semiconductor laser with optical feedback

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Abstract

We numerically study the effect of additive Gaussian white noise in the dynamics of a time-delayed feedback system. The system is a semiconductor laser with optical feedback from a distant reflector. For moderate feedback levels the system presents several coexisting attractors, and noise levels above a threshold value induce jumps among these attractors. Based on the residence times probability density, $P(I)$, we show that with increasing noise the dynamics of attractor jumping exhibits a resonant behavior. $P(I)$ presents peaks at multiples of the external-cavity delay time, and the strength of the peaks reaches a maximum value for an optimal level of noise. The results are explained by the interplay of noise and delayed feedback.

Keywords: Noise; Time delay; Semiconductor lasers; Chaos

It is well known that in nonlinear systems an adequate amount of noise may induce a more ordered behavior. Examples of the constructive role of noise are the enhancement of the response of a bistable system to a weak periodic forcing signal (*stochastic resonance*) [1], the appearance of regular pulses in an excitable system (*coherence resonance*) [2], the formation of patterns in spatially extended systems (*noise-sustained spatial structures*) [3]; the decay from an unstable state driven by amplification of noise (for example, the laser switch-on [4]), among others. Many of these effects have been recently reviewed by San Miguel and Toral [5].

In a previous work we have shown that when a system has a time-delayed feedback loop, noise might induce a new resonance phenomenon [6]. In this paper we study with detail this resonance phenomenon in the dynamics of a semiconductor laser with optical feedback from a distant external reflector. For moderate feedback from a distant reflector, the system exhibits the coexistence of several chaotic attractors. We find a resonant behavior in the noise-induced attractor jumps, which is measured based on the probability density of the residence time in an attractor, $P(I)$. $P(I)$ exhibits a structure of peaks at multiples of the external-cavity delay time, and the strength of the peaks reaches a maximum value for an optimal level of noise. The results are interpreted in terms of the interplay of noise and the delayed feedback loop.

The model equations are as follows [7]:

$$\dot{E} = k(1+\mathrm{i}\alpha)[G-1]E(t) + \gamma E(t-\tau)\,\mathrm{e}^{-\mathrm{i}\omega_0\tau} + \sqrt{D}\xi(t), \tag{1}$$

$$\dot{N} = \frac{j - N - G|E|^2}{\tau_n}. \tag{2}$$

Here, E is the slowly varying complex field, and N is the normalized carrier density. The parameters are: k is the cavity losses, α the linewidth enhancement

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factor, $G = N/(1 + \epsilon|E|^2)$ the optical gain (where ϵ is the gain saturation coefficient), γ the feedback intensity, ω_0 the optical frequency without feedback, and τ is the round-trip time in the external cavity. j is the normalized injection current, and τ_n is the carrier lifetime. $\xi(t)$ is a complex Gaussian white noise, and D measures the noise intensity. Since the resonance effect reported in this paper occurs for large noise levels (larger than the typical value of the rate of spontaneous emission), we consider a noise term that represents externally injected incoherent light (and therefore, the noise term does not depend on the number of carriers in the active medium). Although the deterministic model does not include multiple longitudinal modes, or multiple reflections in the external cavity, it describes qualitatively well many of the observed phenomena, over a wide range of parameter values [8].

The fixed points of (1) and (2) can be written as $E(t) = E_i \exp[\mathrm{i}(\omega_i - \omega_0)t]$ and $N(t) = N_i$, respectively. The frequencies ω_i are the solutions of

$$\omega\tau - \omega_0\tau + \gamma\tau\sqrt{1+\alpha^2}\sin(\omega\tau + \arctan\alpha) = 0, \quad (3)$$

and are shifted positively and negatively with respect to ω_0. The carrier density and field amplitude of a fixed point are functions of ω_i. For increasing feedback pairs of fixed points appear though saddle-node bifurcations, and each initially stable fixed point undergoes a quasiperiodic route to chaos. The two frequencies that appear in the route to chaos are the same for all the fixed points. One is nearly equal to the frequency of the relaxation oscillations of the solitary laser, f_{ro}, and the other is nearly equal to the external-cavity frequency, $f_{\mathrm{ext}} = 1/\tau$.

The chaotic attractors are localized in phase space around the destabilized fixed points. If the feedback level is not too large, the coexisting attractors are widely separated in phase space. Higher feedback increases the volume of the attractors, inducing a *deterministic* merging of the ruins of several attractors to form a global attractor. In this global attractor the trajectory traverses various 'attractor ruins', spending a certain amount of time in the vicinity of an attractor, before jumping to another. The jumps are self-triggered by the dynamics, without the need for the presence of external noise or parameter variations. The dynamics is a form of *chaotic itinerancy* [9], which consists of the successive visit of different manifolds (with a chaotic dynamics within each of them), persisting for a time much longer than the transition time from one another. For low injection current and moderate feedback the laser exhibits excitable behavior, with random, abrupt, intensity dropouts followed by gradual, deterministic recoveries [10]. The dynamics of Eqs. (1) and (2) in this regime is a form of *chaotic itinerancy with a drift* [11], where the trajectory traverses the attractor ruins with a definite direction in phase space towards the fixed point with maximum gain.

Here we chose parameters such that the feedback level is slightly lower than the feedback at which *deterministic* attractor merging begins. In the absence of noise, which attractor the trajectory evolves to in its long-term behavior depends only on the initial conditions. Noise levels above a threshold value induce jumps among the attractors (*stochastic* attractor merging). The injection current is chosen large enough such that the jumps are random in direction.

Fig. 1(a) shows a typical stochastic trajectory, plotted in the plane formed by $(\Delta\phi, |E|^2)$, where $\Delta\phi = \phi(t) - \phi(t-\tau)$ is proportional to the laser frequency averaged over a time τ (ϕ is the phase of the complex field E), and $|E|^2$ is proportional to the laser intensity. Fig. 1(b) illustrates the time evolution of the phase delay, where the attractor jumps can be clearly distinguished.

The effect of noise on a stochastic trajectory is shown in Fig. 2. For low noise (Fig. 2(a)), the trajectory might spend a large amount of time in an attractor before noise induces a jump. Larger noise levels induce more frequent jumps (Fig. 2(b) and (c)), and the mean residence time on an attractor diminishes. For even larger noise the dynamics becomes increasingly noisy, until there is almost no structure present in the trajectory.

Fig. 3 shows the power spectrum of $\Delta\phi(t)$, for the same parameters and noise levels as Fig. 2. The frequencies appearing in the spectra are the relaxation oscillation frequency, $f_{\mathrm{ro}} \sim 3.5\,\mathrm{GHz}$, and the external-cavity frequency, $f_{\mathrm{ext}} \sim 0.1\,\mathrm{GHz}$, and their harmonics. The investigation of the spectra was

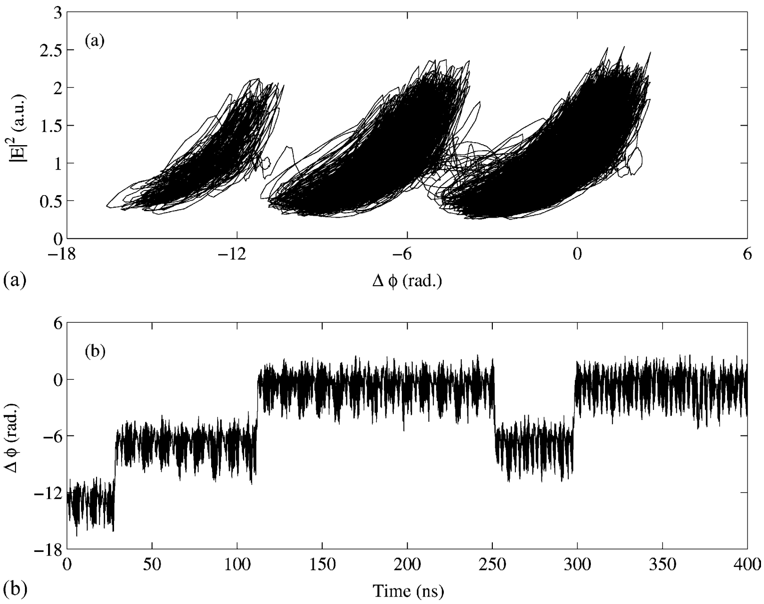

Fig. 1. (a) Global attractor created by noise-induced jumps. (b) Time evolution of the phase delay. The parameters are $k = 500\,\mathrm{ns}^{-1}$, $\tau_n = 1\,\mathrm{ns}$, $\alpha = 3$, $j = 2$, $\epsilon = 0.003$, $\gamma = 2\,\mathrm{ns}^{-1}$, $\tau = 10\,\mathrm{ns}$, $\omega_0\tau = 6\,\mathrm{rad}$, and $D = 0.0225\,\mathrm{ns}^{-1}$.

motivated by the conjecture put forward by Arecchi and co-workers [12], that a multistable system with fractal basin boundaries disturbed by noise (such that attractor jumping occurs), exhibits a $1/f^{\alpha}$ spectrum at low frequencies, with $\alpha \sim 0.5$–1.5. The exact conditions that give rise to a low-frequency spectral divergence are still not fully clarified [13], but experiments and simulations suggest that some 'weak stability' in

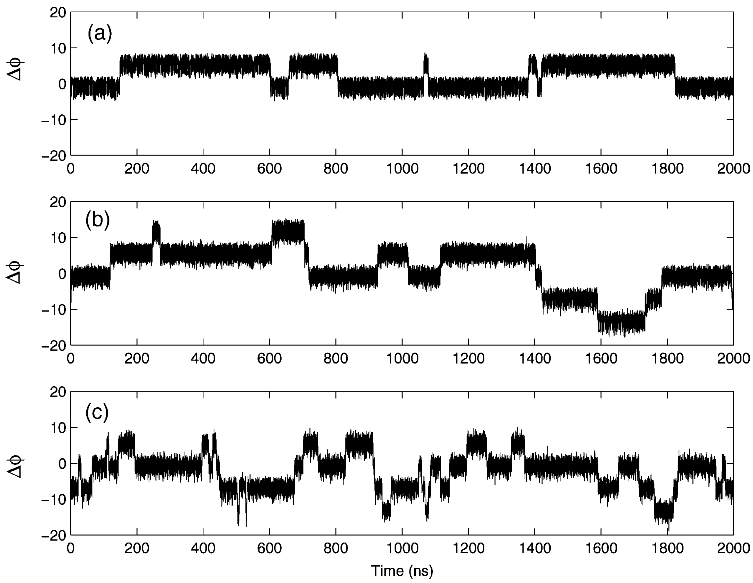

Fig. 2. Typical stochastic trajectory: (a) $D = 0.0025\,\mathrm{ns}^{-1}$; (b) $D = 0.0225\,\mathrm{ns}^{-1}$; (c) $D = 0.09\,\mathrm{ns}^{-1}$. All other parameters as in Fig. 1.

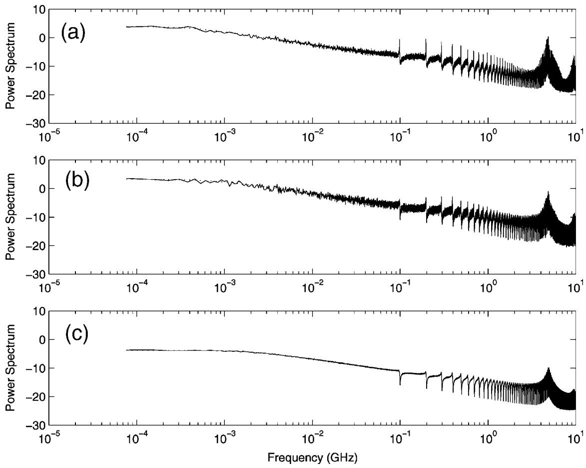

Fig. 3. Power spectrum of $\Delta\phi(t)$, for the same parameters and noise levels as in Fig. 2. The spectra were calculated averaging over 100 trajectories with different noise realizations.

the attractors is required. A $1/f^{\alpha}$ spectrum was found in experiments with electronic circuits and lasers [12], and more recently, in simulations of a multiattractor map [14].

The spectra shown in Fig. 3 are all well fit by a Lorentzian, with a flat low-frequency part, and a $1/f^2$ high-frequency asymptotic tail. A nontrivial low-frequency spectrum was not observed, perhaps because for the feedback level considered the attractors of the noiseless system are stable. For a slightly higher feedback level (for which there is deterministic attractor merging), the spectrum of $\Delta\phi(t)$ (illustrated in Fig. 4) changes quantitatively, being well fit, at low frequencies, by a power law $1/f^{\alpha}$ with $\alpha \sim 1.3$.

Next, the statistics of the residence time in an attractor is investigated as a function of the noise intensity. The time interval between two consecutive jumps was determined by approximating the jumps as instantaneous events. To determine the time at which a jump occurs, at every step of the integration the value of $\Delta\phi$ was averaged over a time window of a few oscillation periods (typically, one to three oscillation periods). If the trajectory is in the vicinity of the ith fixed point, $\langle\Delta\phi\rangle \sim (\omega_i - \omega_0)\tau$. Therefore, at each step of the integration the label 'i' of the attractor was determined, and comparing with the label in the previous step, it was established if a jump occurred. The time window over which the phase delay was averaged had to be short enough to accurately detect the fast jumps, but not too short to detect 'false jumps', due to the oscillations of $\Delta\phi(t)$ when the trajectory evolves in the vicinity of an attractor.

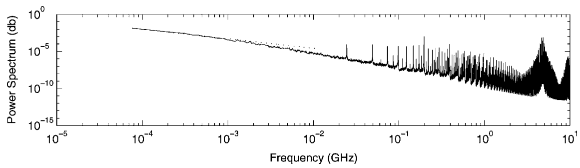

Fig. 4. Power spectrum of $\Delta\phi(t)$ when deterministic attractor merging begins. The parameters are $\gamma = 2.2\,\mathrm{ns}^{-1}$, $D = 0$ and the rest are as in Fig. 3.

To obtain a good statistics, we had to compute a large number of jumps for each noise level. We integrated N_1 trajectories (with the same parameter values, initial conditions, and noise intensity, but distinct noise realizations), a time long enough such that in each trajectory, N_2 jumps occurred. In this way, we obtained a total of N_1N_2 jumps for each noise level. Since the jumps are more frequent for large noise, for low noise it was necessary to integrate the trajectory a long time, to obtain the desired number of jumps. In the following, the results shown are evaluated from samples of more than 10^6 jumps.

Fig. 5 shows the probability distribution, $P(I)$, of residence times for increasing noise. $P(I)$ decreases exponentially for large residence times (for low noise there is a large spread in the values of I, and the exponential decay for large I is not seen in Fig. 5(a)). For short residence times, $P(I)$ exhibits a multipeaked structure which depends on the noise level. On increasing the noise level from small up to large values, $P(I)$ goes from a distribution with minimums (or 'gaps') for values of I that are multiple of τ, to a distribution with maximums for values of I that are multiple of τ. The peaks are superimposed onto an exponentially decaying background, which is weak for low noise, but that grows and hides the peak structure for large noise.

For low noise, the gaps of $P(I)$ at $I \sim n\tau$ are due to the deterministic dynamics on an attractor and can be understood by considering the moment when the trajectory can jump from one attractor to another. In a single attractor the trajectory spends some time

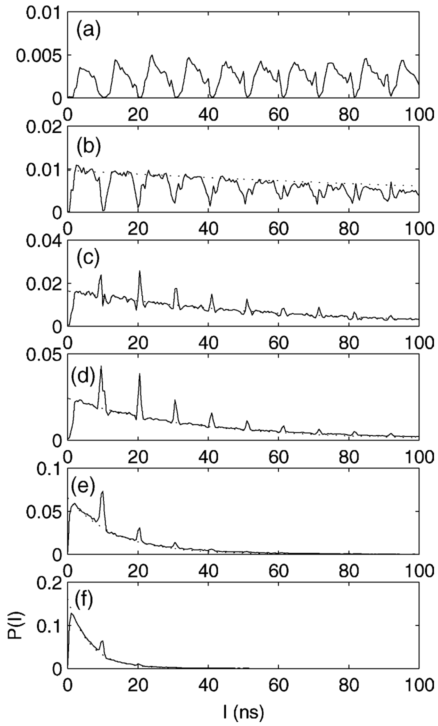

Fig. 5. Probability distribution of short residence times: (a) $D = 0.0025\,\mathrm{ns}^{-1}$; (b) $D = 0.0225\,\mathrm{ns}^{-1}$; (c) $D = 0.04\,\mathrm{ns}^{-1}$; (d) $D = 0.09\,\mathrm{ns}^{-1}$; (e) $D = 0.16\,\mathrm{ns}^{-1}$; (f) $D = 0.25\,\mathrm{ns}^{-1}$. All other parameters as in Fig. 1.

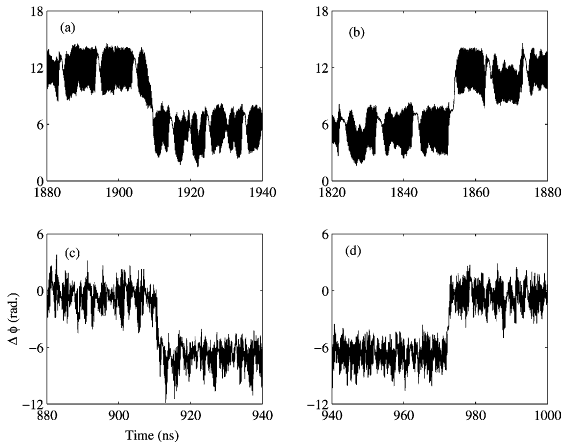

Fig. 6. Typical attractor jumps: (a), (b) $D = 0.0025\,\mathrm{ns}^{-1}$; (c), (d) $D = 0.09\,\mathrm{ns}^{-1}$.

orbiting around the destabilized fixed point, until it reaches the neighborhood of the fixed point. Then, the trajectory traverses the vicinity of the fixed point, and starts oscillating again. This process, in which the trajectory is orbiting around the fixed point, or is in the vicinity of the fixed point, keeps repeating with a period $\sim\tau$. The noise-induced jump to an attractor with lower $\Delta\phi$ usually occurs in the middle of the stage in which the trajectory oscillates around the fixed point (Fig. 6(a)), while the jump to an attractor with larger $\Delta\phi$ occurs at the end of the oscillating stage (Fig. 6(b)). Therefore, for low noise, residence times that are multiple of τ are less probable.

For larger noise, the deterministic dynamics on an attractor is mostly washed out by the noise, and a jump can occur at any time (Fig. 6(c) and (d)). For even larger noise, the residence times $I \sim n\tau$ become increasingly probable. This unexpected feature is caused by the delayed feedback loop, and can be understood qualitatively in the following terms: a fluctuation strong enough to trigger a jump, due to the delay term in (1) is re-injected in the system and might induce another jump, one or few delay times later. Therefore, with increasing noise residence times multiple of τ become increasingly probable.

Similar peaks in $P(I)$ are found in stochastic resonance [15], and the strength of the peaks (once the exponential background was subtracted) is used to quantify the resonance. Fig. 7 shows that the strength of the nth peak achieves a maximum at a certain level of noise, D_{opt}. D_{opt} decreases with the index n of the peak, and this can be interpreted in the following

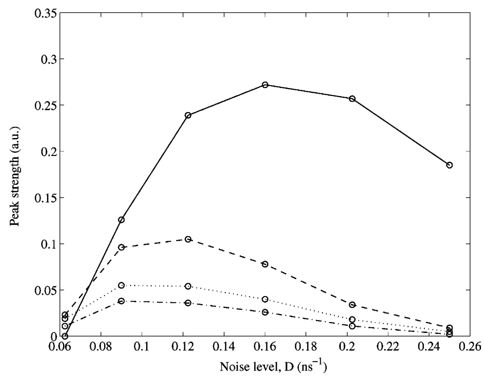

Fig. 7. Peak strengths P_1 (solid), P_2 (dashed), P_3 (dotted), and P_4 (dash-dotted), as a function of D. The parameters are as in Fig. 1.

terms: the mean residence time $\langle I\rangle$ diminishes with increasing noise, and the trajectory does not stay in an attractor long enough to jump out of it due to the delayed effect of the fluctuation that induced the original jump.

In conclusion, the effect of noise was studied numerically in a semiconductor laser with optical feedback from a distant reflector. The major finding of the analysis is a resonant behavior in the noise-induced attractor jumps, which is measured by the residence times probability density. $P(I)$ exhibits a structure of peaks at multiples of the delay time, and strength of the peaks reaches a maximum for an optimal level of noise. These results are interpreted in terms of the interplay of noise and the time-delayed feedback loop.

It is important to remark the difference between the resonance effect reported in this paper and well-known phenomenon of coherence resonance (see, e.g., [2,16]). The resonance reported here is due to the existence of a feedback loop with a characteristic delay time, which is absent in the physical mechanism of coherence resonance.

Time-delayed systems are infinite dimensional systems, and there are important physical and biological models with time delays. Our results suggest that time-delayed dynamics, typical for instance in neuronal processes, might introduce noise-induced phenomena not found in simpler models of such systems.

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