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Physica A 371 (2006) 104–107
PHYSICA A
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Chaotic maps coupled with random delays: Connectivity, topology, and network propensity for synchronization

Arturo C. Martí[a,*], Marcelo Ponce C.[a], Cristina Masoller[a,b]

[a]*Instituto de Física, Facultad de Ciencias, Universidad de la República, Iguá 4225, Montevideo 11400, Uruguay*
[b]*Departament de Fisica i Enginyeria Nuclear, Universitat Politecnica de Catalunya, Colom 11, E-08222 Terrassa, Spain*

Available online 19 May 2006

Abstract

We study the influence of network topology and connectivity on the synchronization properties of chaotic logistic maps, interacting with random delay times. Four different types of topologies are investigated: two regular (a ring-type and a ring-type with a central node) and two random (free-scale Barabasi–Albert and small-world Newman–Watts). The influence of the network connectivity is studied by varying the average number of links per node, while keeping constant the total input that each map receives from its neighbors. For weak coupling, the array does not synchronize regardless the topology or connectivity of the network; however, for certain connectivity values there is enhanced coherence. For strong coupling, the array synchronizes in the homogeneous steady-state, where the chaotic dynamics of the individual maps is suppressed. For both, weak and strong coupling, the array propensity for synchronization is largely independent of the network topology and depends mainly on the average number of links per node.

Keywords: Complex networks; Random delays; Logistic map; Coupled map lattices

The emergence of dynamical order in complex systems has been widely studied during the last years [1–3]. One important issue is the propensity for synchronization of networks of dynamical elements. In this context, coupled maps [4] are excellent tools for understanding the mechanisms of emergency of synchrony and collective behavior in complex systems composed of mutually coupled nonlinear units. Not only from an academical point of view but also from an applied perspective, cooperative behavior arises in many fields of science and classical examples include the onset of rhythmic activity in the brain, the flashing on and off in unison of populations of fireflies, the emission of chirps by populations of crickets and the synchronization of laser arrays and Josephson junctions [1]. Coupled map lattices have proven to be a useful tool because by simplifying the dynamics of the individual units it is possible to simulate large ensembles of coupled units.

Network synchronizability and its relation with the topology has recently received a great deal of attention. Atay et al. [5] found, in the case of fixed (constants) delays that scale-free and random networks exhibit better synchronization properties than regular networks. More recently, Motter et al. [6] identified that the synchronization of complex networks follows a diffusive mechanism where the mean connectivity plays a key

*Corresponding author.
E-mail addresses: marti@fisica.edu.uy (A.C. Martí), mponce@fisica.edu.uy (M. Ponce C.), cris@fisica.edu.uy (C. Masoller).

0378-4371/$ - see front matter
doi:10.1016/j.physa.2006.04.093

role. In a previous paper [7], we investigated the relation of the topology with the ability to synchronize under the presence of random delays. We found that in this case the synchronization properties depend largely on the mean connectivity of the network. However, the topology does not play an important role. The aim of this paper is to further investigate this point.

The evolution equations for N coupled logistic map with random delays are

$$x_i(t+1) = (1-\varepsilon)f[x_i(t)] + \frac{\varepsilon}{b_i}\sum_{j=1}^{N}\eta_{ij}f[x_j(t-\tau_{ij})]. \tag{1}$$

Here t is a discrete time index, i is a discrete spatial index ($i = 1 \ldots N$), $f(x) = ax(1-x)$ is the logistic map, the matrix $\eta = (\eta_{ij})$ defines the connectivity of the array: $\eta_{ij} = \eta_{ji} = 1$ if there is a link between the ith and jth nodes, and zero otherwise. ε is the coupling strength and τ_{ij} is the delay time in the interaction between the ith and jth nodes (the delay times τ_{ij} and τ_{ji} need not be equal). The sum in Eq. (1) runs over the b_i nodes which are coupled to the ith node ($b_i = \sum_j \eta_{ij}$). The normalized pre-factor $1/b_i$ means that each map receives the same total input from its neighbors.

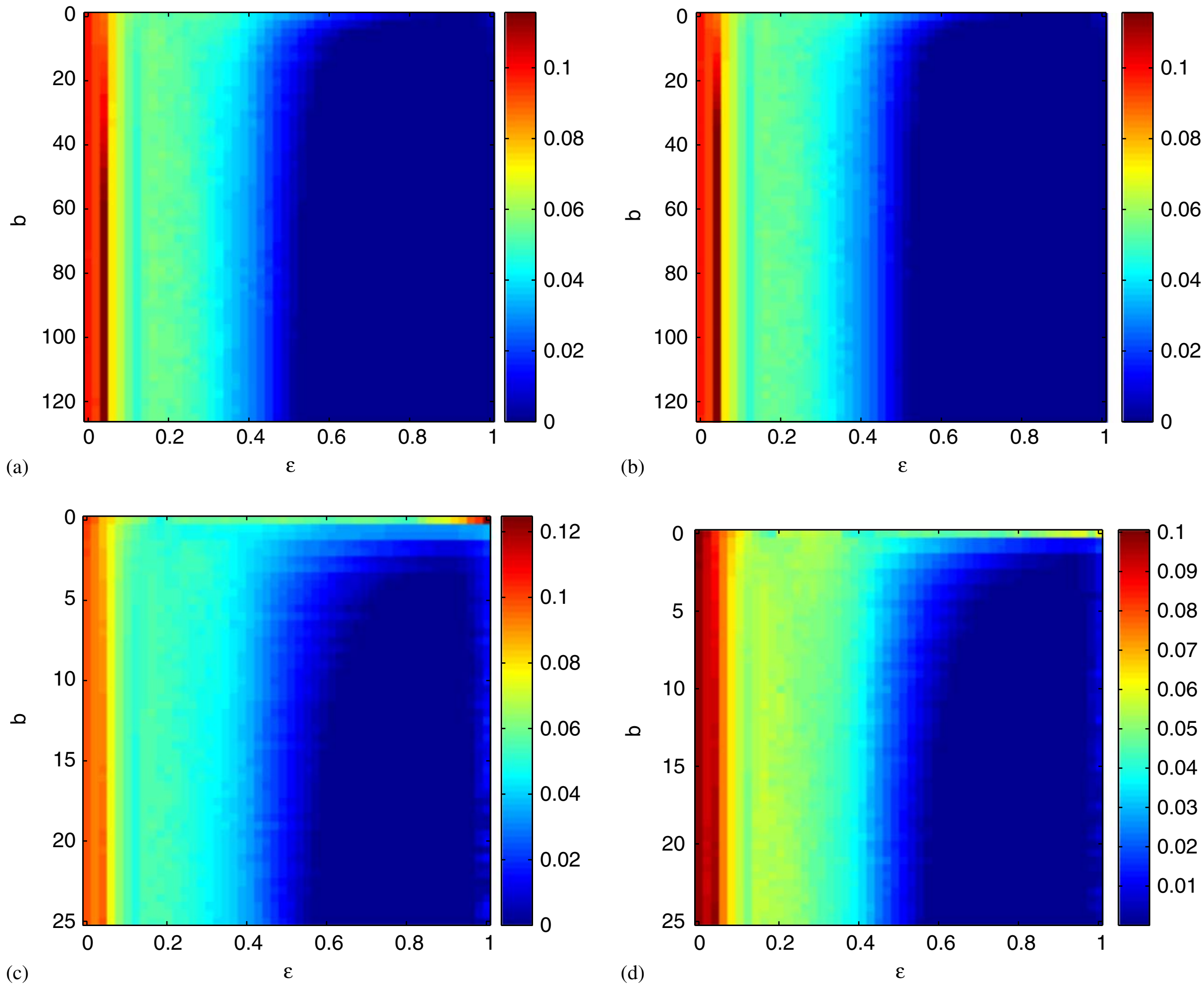

Fig. 1. (Color online) Synchronization regions for the four different networks considered. The density plots represent the parameter σ^2 as a function of ε and b ($N = 500$ and $a = 4$.). (a) Smallworld; (b) scale-free; (c) nearest-neighbors and (d) nearest-neighbor with central node.

In a previous work [7] we found that if the delays τ_{ij} are random enough, for adequate coupling strength the array synchronizes in the *spatially homogeneous steady-state*, $x_i(t) = x_0$ for all i, where x_0 is the non-trivial fixed point, $x_0 = 1 - 1/a$ [7]. This synchronization behavior is in contrast with the behavior with fixed delays (if $\tau_{ij} = \tau_0\ \forall i, j$, the array synchronizes in a spatially homogeneous *time-dependent state*, where the dynamics is either periodic or chaotic depending on τ_0 [5]), and can be understood in terms of the analogy between globally coupled maps and a single map with a external driving [8,9].

To investigate the influence of the topology we consider four networks, two of them are regular and the other two are random. The regular ones are a ring of nearest-neighbor elements while in the second one we added a central node connected to all other nodes. The random networks consist of a scale-free network constructed according to the Barabasi–Albert method and, concerning the last one, we use the small-world topology proposed by Newman and Watts. To characterize the transition to synchronization we use the indicator $\sigma^2 = 1/N\langle\sum_i [x_i(t) - \langle x\rangle]^2\rangle_t$, where $\langle\cdot\rangle$ denotes an average over the elements of the array and $\langle\cdot\rangle_t$ denotes an average over time. If the array synchronizes in a spatially homogeneous state, $x_i(t) = x_j(t)\ \forall i, j$, and, obviously, $\sigma^2 = 0$.

We consider Gaussian distributed delays: $\tau_{ij} = \tau_0 + \text{near}(c\xi)$, where c is a parameter that allows varying the width of the delay distribution; ξ is Gaussian distributed with zero mean and standard deviation one; near denotes the nearest integer. Depending on τ_0 and c the distribution of delays has to be truncated to avoid negative delays. Since the focus of this paper is the influence of the array topology and the connectivity we keep the random delays Gaussian distributed with $\langle\tau_{ij}\rangle \sim \tau_0 = 5$ and $c = 2$. The numerical results are summarized in Figs. 1–3.

In Fig. 1 we can see density plots of σ^2 as a function of the mean number of links per node, $b = 1/N\sum_{i=1}^{N} b_i$ and ε. The four different panels correspond to the different networks mentioned above. Despite the differences for small number of neighbors, we observe that the synchronizability is largely independent of the topology. Furthermore, the similarity between Figs. 1(c) and (d) clearly suggest that the synchronizability does not depend on the average path length (defined as the distance between two nodes, averaged over all pairs of nodes).

For weak coupling (roughly speaking, $\varepsilon \lesssim 0.4$), the array does not synchronize regardless the number of neighbors and topology. However, there are worth noting different behaviors depending on the value of ε. To gain additional insight, it is shown in Fig. 2 the value of σ^2 as a function of b for different values of ε. We observe that for $\varepsilon = 0.02$ (a) the value of σ^2 decreases monotonously as b increases, to a limiting non-zero value. For $\varepsilon = 0.04$ (b) there is still no synchronization for any connectivity value, but there is a non-monotonous dependence of σ^2 with b, that reveals the existence of an optimal number of neighbors for which

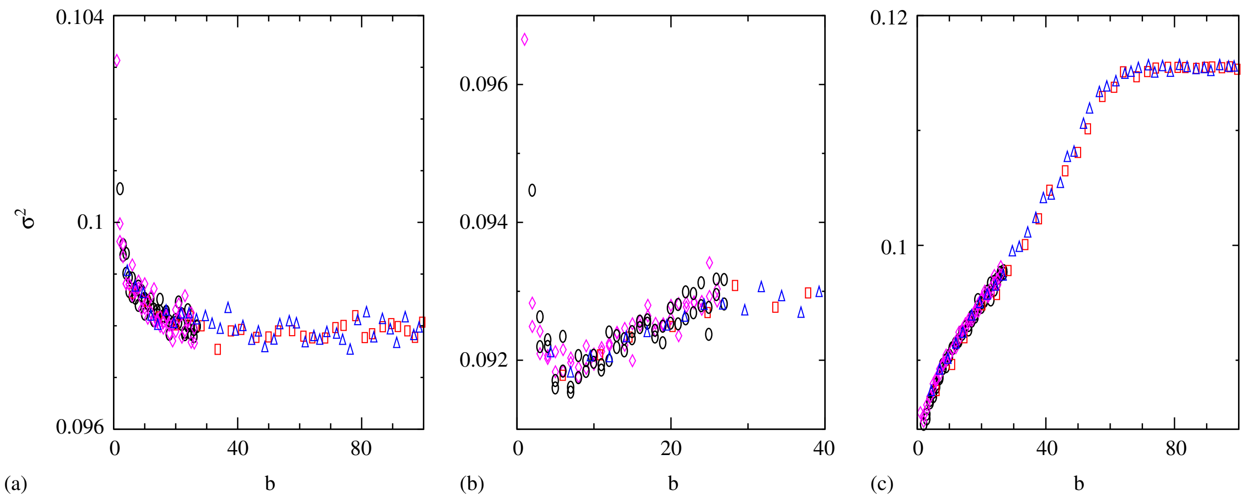

Fig. 2. σ^2 as a function of the number of neighbors b for the weak coupling regime and different topologies: nearest-neighbors with central node (○); scale-free (□); nearest-neighbors (◇) and small-world (△). Parameters are: $N = 500$, $a = 4$; (a) $\varepsilon = 0.02$; (b) $\varepsilon = 0.04$ and (c) $\varepsilon = 0.06$.

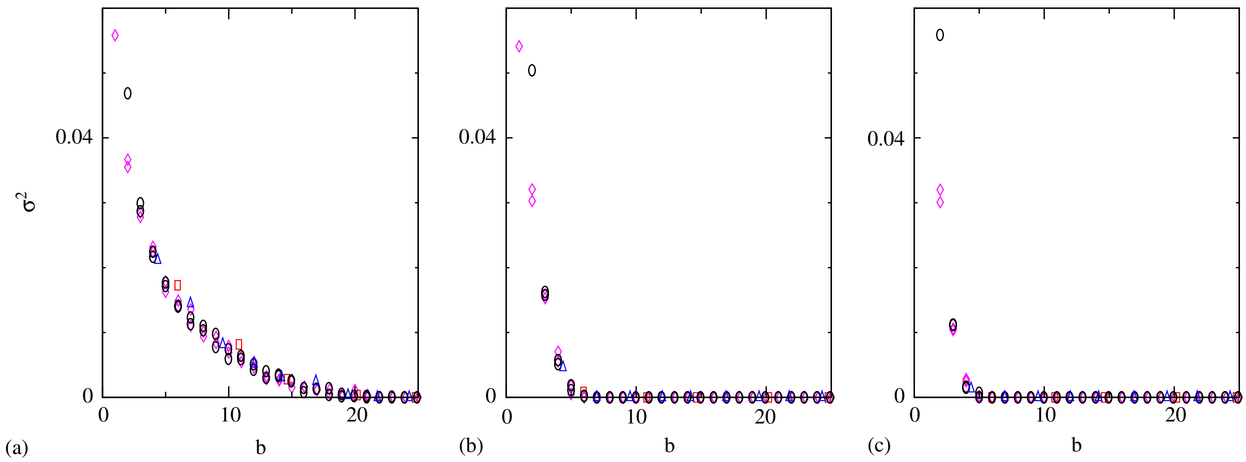

Fig. 3. σ^2 as a function of the number of neighbors for the strong coupling regime and different topologies. (a) $\varepsilon = 0.6$, (b) $\varepsilon = 0.8$ and (c) $\varepsilon = 0.9$. Same symbols and parameters values as Fig. 2.

there is an enhancement of the array propensity for synchronization. Finally, in (c) we observe that, for $\varepsilon = 0.06$, in sharp contrast to (a), σ^2 increases monotonously before reaching a non-zero limiting value.

On the other hand, for strong enough coupling ($\varepsilon \gtrsim 0.4$) the array synchronizes in the homogeneous steady state if the number of the neighbors is large enough. In Fig. 3 we observe σ^2 as functions of b for different values of ε. Depending on the value of ε there is a minimum value of neighbors required to synchronize the array. Moreover, this critical value decreases with increasing ε.

In summary, we studied the synchronization of coupled maps in complex networks with time-delayed interactions focusing on the influence of array connectivity and topology. For weak coupling no synchronization was found, but an enhancement of the synchronization propensity was observed for particular connectivity values. On the other hand, for strong coupling there is synchronization provided that the number of neighbors was large enough.

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