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The role of intermediaries in the synchronization of pulse-coupled oscillators

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Abstract. The role of intermediaries in the synchronization of small groups of light controlled oscillators (LCO) is addressed. A single LCO is a two-time-scale phase oscillator. When pulse-coupling two LCOs, the synchronization time decreases monotonously as the coupling strength increases, independent of the initial conditions and frequency detuning. In this work we study numerically the effects that a third LCO induces to the collective behavior of the system. We analyze the new system by dealing with directed heterogeneous couplings among the units. We report a novel and robust phenomenon, absent when coupling two LCOs, which consists of a discontinuous relationship between the synchronization time and coupling strength or initial conditions. The mechanism responsible for the appearance of such discontinuities is discussed.

1 Introduction

Collective behavior, such as synchronization [1], is ubiquitous in nature. In particular, a paradigmatic example of synchronous behavior is observed in gregarious fireflies communities [2]. Initially inspired in the emissions of gregarious fireflies, light controlled oscillators (LCOs) are aimed to study the mechanisms which drive their collective behavior into synchronization [3–7].

An LCO is a relaxation oscillator which interacts with others by means of light pulses. When an LCO is uncoupled, its oscillation is composed of a charging state and a discharging state. Usually, the characteristic time-scales are chosen such that the charging state lasts significantly more than the discharging state (an extreme case is the integrate-and-fire oscillation, where discharge is instantaneous). When considering two or more coupled LCOs, the dynamics of the individual oscillators is modified due to the interaction. The reason is that, during the discharging state, the LCO is able to emit a pulse which affects the LCOs connected to it. The influence of the received pulse over the oscillation is either excitatory (the oscillation is accelerated), if the affected LCO is at its charging state, or inhibitory (the oscillation is decelerated), if the affected LCO is at its discharging state.

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One of the advantages of the LCOs is that they are easily implemented using simple electronic devices [3,4] that facilitate the experimental analysis of their transient times and emergent properties. The transient time needed to synchronize, i.e., the synchronization time, is of crucial interest in several technological fields. In general, when two self-sustained oscillators are coupled, the synchronization time decreases as the coupling strength is increased. In particular, this behavior is also observed between two interacting LCOs. However, in the special case of bidirectional coupling between two LCOs, a critical coupling strength value exists beyond which any higher value of coupling destroys the oscillation. Such a novel phenomenon is known as oscillation death [8–10].

In the case when the interaction between two LCOs is asymmetrical, the oscillators can be ordered according to a well-defined hierarchy. Namely, the oscillator that influences the most is said to have the highest hierarchy. For example, this is the case of a *master-slave* (MS) configuration, where one LCO influences another without being influenced by it. In contrast, when the coupling is bidirectional, with no higher or lower hierarchy, the system is said to be in a *mutual interaction* (MI) configuration. Ref. [11] exploides a parallelism between MS and MI configurations and driven and autonomous maps which display equivalent synchronization characteristics.

When dealing with the task of lowering the synchronization time between two oscillators in MS configuration, a possibility is to add a third oscillator that mediates between the master and the slave, i.e., an intermediate hierarchy. On the other hand, if an intermediate hierarchy is added to the MI configuration, the directionality of the added links destroys the equality in hierarchies of the original LCOs configuration and the resultant configuration is a non-trivial hierarchical relationship.

The study of synchronization in systems of three oscillators has been addressed in previous works, for example, relay coupling [12], environmental coupling [13], or chemical oscillators [14]. In those cases, the oscillators are arranged in a three-on-a-row configuration and, as a result, the interaction between the *master* and the *slave* is given exclusively through the intermediary. While in those cases the equations of motion are continuous and have continuous derivatives, we deal here with piecewise analytical equations of motion, which introduce singularities in the derivatives.

In this work we study the influence of adding an intermediary oscillator to the synchronization times of two pulse-coupled oscillators using various coupling configurations. Our numerical experiments show the appearance of a novel phenomenon in the relationship between the coupling strength and synchronization time. Contrary to the monotonous decrease that the synchronization times of two oscillators exhibit as the coupling strength increases, the novel phenomenon is accounted by a discontinuous relationship between synchronization times and coupling strengths and is solely explained due to the inclusion of the intermediary. Moreover, we derive analytically the conditions the intermediary oscillator must fulfil in order to reduce the transient times towards synchronization.

2 Model

The equation describing the evolution of the i-th coupled LCO is [3,5]

$$\frac{dV_i(t)}{dt} = [E_i(t) - \lambda_i(V_i(t) - V_{cc})]\,\epsilon_i + [E_i(t) - \gamma_i V_i(t)](1 - \epsilon_i), \qquad (1)$$

where $V_i(t)$ is the oscillating state variable of the i-th oscillator at time t, ϵ_i is a binary value that represents the state in which the i-th oscillator is, taking the value 1 for the *charging state* and 0 for the *discharging state*, the parameter V_{cc} determines
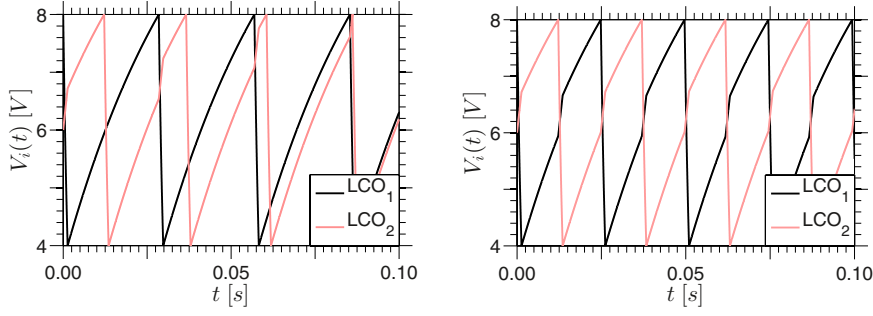
Fig. 1. Evolution window of the state variable $V_i(t)$ of two LCOs coupled in a MS configuration (left) and MI configuration (right). In both panels the oscillators are identical, with $T_\lambda = 27.8$ ms and $T_\gamma = 0.7$ ms, and the coupling strength $\beta = 650$ V/s.

the oscillation amplitude, and $E_i(t)$ is the coupling. The state of every LCO changes from charge (discharge) to discharge (charge), $\epsilon_i = 1 \rightarrow 0$ ($\epsilon_i = 0 \rightarrow 1$), when its state variable, $V_i(t)$, reaches $2V_{cc}/3$ ($V_{cc}/3$). The parameters λ_i and γ_i define the characteristic charging and discharging state frequencies, respectively. Hence, the i-th LCO natural (uncoupled) frequency is

$$\nu_i = (T_{\lambda_i} + T_{\gamma_i})^{-1} = \left(\frac{\log 2}{\lambda_i} + \frac{\log 2}{\gamma_i} \right)^{-1}. \tag{2}$$

The choice of notation in Eq. (1) and the dimensional values for the LCOs (units of voltage for $V_i(t)$, frequency for λ_i and γ_i, and units of voltage per second for the $E_i(t)$) are inspired in the experimental implementation of the system [4]. This experimental setup is based on a dual RC circuit, with a charging state given by the characteristics of one of the RC and a discharging state given by the characteristics of the second RC circuit.

The coupling term is explicitly given by

$$E_i(t) = \sum_{j \neq i} a_{ij} \beta_{ij} (1 - \epsilon_j), \tag{3}$$

a_{ij} being the adjacency matrix entry corresponding to the possible connection between nodes i (LCO$_i$) and j (LCO$_j$), namely, $a_{ij} = 1$ if the LCOs are connected and $a_{ij} = 0$ otherwise, and β_{ij} being the coupling strength of the particular connection. The state modification that is induced by this coupling function results in an acceleration of the charging state (excitation) and/or a deceleration for the discharging state (inhibition). Asymmetry in the couplings is achieved by setting $\beta_{ij} \neq \beta_{ji}$, though the adjacency matrix is assumed to be always symmetrical. For example, an MS configuration between LCOs i and j is achieved by setting $\beta_{ij} > 0$ and $\beta_{ji} = 0$.

Although Eq. (1) is a piecewise linear coupled set of differential equations, a general and unique analytical solution, which is piecewise smooth, exists for each initial condition [5]. In particular, a solution for two coupled LCOs in a MS (left panel) and MI (right panel) configuration are shown in Fig. 1 using the same initial conditions ($V_1(0) = 2V_{cc}/3$ discharging and $V_2(0) = V_{cc}/2$ charging, with $V_{cc} = 12$ V). We note that the initial condition choice represents the unstable anti-synchronous state of the MI configuration (i.e., both oscillators perform the same oscillation but with a phase difference of π), hence, the behavior of the transient time-window shown in the right panel of Fig. 1.
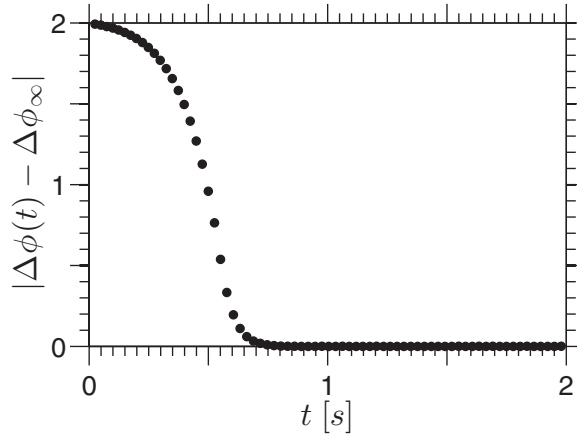
Fig. 2. Phase difference evolution towards synchronization between a *master* LCO and a *slave* LCO in the interaction between two oscillators. The absolute value of the difference between the phase difference and the final phase difference decreases to zero, and it remains constant when the synchronous state is reached. In this case the synchronization time is 804.8 ms.

As each LCO is a one dimensional oscillator, piecewise linear, a phase $\phi_i(V_i)$ is defined using the free oscillation frequency of each LCO. Specifically,

$$\phi_i(V_i) = \begin{cases} 2\pi \left(\dfrac{-1}{\log 2} \dfrac{\gamma_i}{\lambda_i + \gamma_i} \log \left[\dfrac{3}{2} \left(1 - \dfrac{V_i(t)}{V_{cc}} \right) \right] \right), & \epsilon = 1, \\[4mm] 2\pi \left(1 - \dfrac{1}{\log 2} \dfrac{\lambda_i}{\lambda_i + \gamma_i} \log \left[3 \dfrac{V_i(t)}{V_{cc}} \right] \right), & \epsilon = 0. \end{cases} \tag{4}$$

Equation (4) allows to determine the phase difference evolution, namely, $\Delta\phi(t)$, between LCOs. This provides a quantitative measure of the system's emergent properties. For the synchronization, it is expected for it to converge to a constant value (zero if the system has complete synchronization [15] and finite for phase synchronization [16]). Hence, in order to calculate the synchronization times, the difference, $\Delta(t)$, between $\Delta\phi(t)$ and its limit value, $\Delta\phi_\infty = \lim_{t\to\infty} \Delta\phi(t)$, is defined. For example, Fig. 2 shows how $\Delta(t)$ decreases as the system evolves towards the synchronized state, regardless if it is a complete synchronous ($\Delta\phi_\infty = 0$) state or a phase synchronous ($\Delta\phi_\infty > 0$) state. In both cases, $\lim_{t\to\infty} \Delta(t) = 0$.

3 Results

3.1 Two-LCO dynamics

We start the study of synchronization times restricting ourselves to the case of two coupled LCOs. The possible hierarchies in such cases are the *master-slave* (MS) configuration (extreme hierarchical difference) and the *mutual interaction* (MI) configuration with symmetrical coupling strengths (identical hierarchy). For the sake of simplicity, we deal with identical oscillators, namely, $\lambda_i = \lambda$, $\gamma_i = \gamma$, and $\beta_{ij} = \beta$, for all LCOs from $i = 1, 2$. In order to construct the synchronization time dependence on the coupling strength, the initial condition of one of the oscillators is set to be
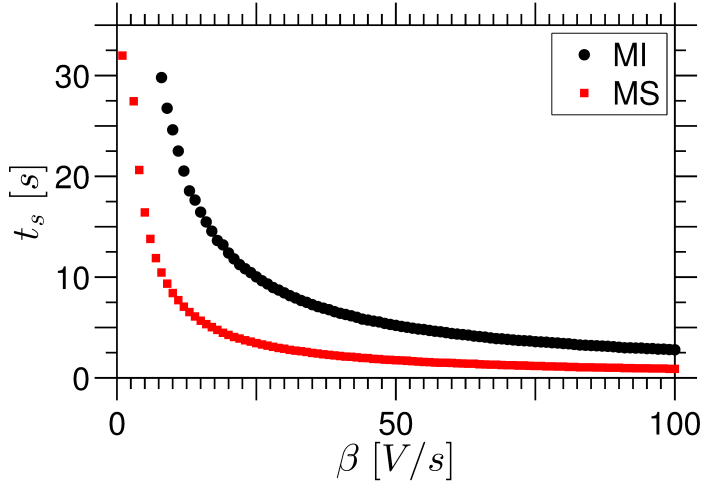
Fig. 3. Synchronization time as a function of the coupling strength for *master-slave* (MS, filled light – red online – squares) and *mutual interaction* (MI, filled dark – black online – circles) configurations. In both cases, the two oscillators are identical, with a period given by $T_\lambda = 27.8$ ms and $T_\gamma = 0.7$ ms [Eq. (2)]. Initial conditions are fixed to $V_1(0) = 2V_{cc}/3$ (the *master* in the MS configuration) and $V_2(0) = V_{cc}/2$ (the *slave* in the MS configuration).

$V_1(0) = 2V_{cc}/3$ (which corresponds to the master LCO in the MS configuration), and the other oscillator is set to be $V_2(0) = V_{cc}/2$.

  We note that, unless the initial conditions for the oscillators are set to be identical $[V_1(0) = V_2(0)]$, the synchronization time, t_s, decreases monotonously as the coupling strength (β) is increased, regardless of the particular choice of initial condition. The reason for such universal behavior to exist is that the system, Eq. (1), is scalable into a non-dimensional form which only depends on the ratio between λ and γ and the coupling strength β [6]. The atypical case, $V_1(0) = V_2(0)$, corresponds to the synchronization manifold [17] of the coupled system when the LCOs have identical parameters (λ and γ), hence, its synchronization time vanishes for any coupling strength. In other words, for any initial condition that avoids the case $V_1(0) = V_2(0)$, we observe an scalable curve for the synchronization times as a function of coupling strength.

  An example of the universal behavior synchronization times exhibit for both configurations, MI (filled dark – black online – circles) and MS (filled light – red online – squares), is shown in Fig. 3. This figure shows that the synchronization time decreases monotonously as the coupling strength increases. Moreover, we observe that, for every given β, a lower synchronization time is achieved in the MS configuration in comparison to the MI configuration for this particular choice of initial conditions. Furthermore, at sufficiently weak coupling strengths ($\beta \lesssim 10$ V/s), a non-synchronous region appears for the MI configuration which is absent in the MS case (left region in Fig. 3). The reason for the existence of such region in the MI case is that the choice of initial conditions place the system in the anti-synchronous manifold, hence, its synchronization time its strictly infinite, though any perturbation draws the system out of this manifold allowing the system to achieve synchrony only at very large times.

  When coupling two non-identical LCOs, a complete synchronization manifold is absent [18]. Nevertheless, we observe a similar universal behavior for the monotonous decrease in the synchronization times as the coupling strength increases. The reason is that the system is able to synchronize with a phase lag, i.e., phase synchronization [1], as long as the parameter difference between the LCOs is small
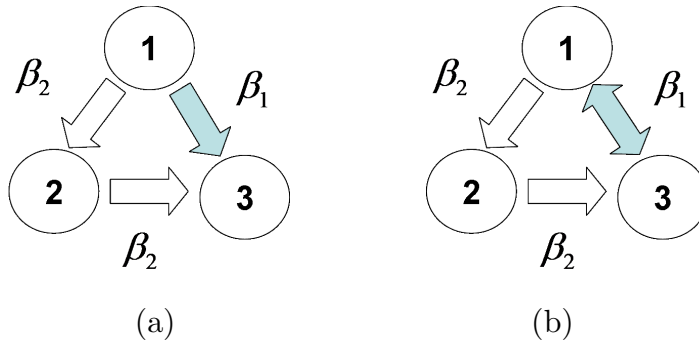
Fig. 4. Schematic diagrams of the master-slave (MS) configuration [oscillators 1 and 2] with an intermediary [oscillator 2] (a) and the mutual interaction (MI) configuration with an intermediary (b). The colored links correspond to the MS and MI without an intermediary oscillator. The coupling strengths involving the intermediary oscillator (LCO_2) are set to be symmetrical to reduce the parameter space dimension.
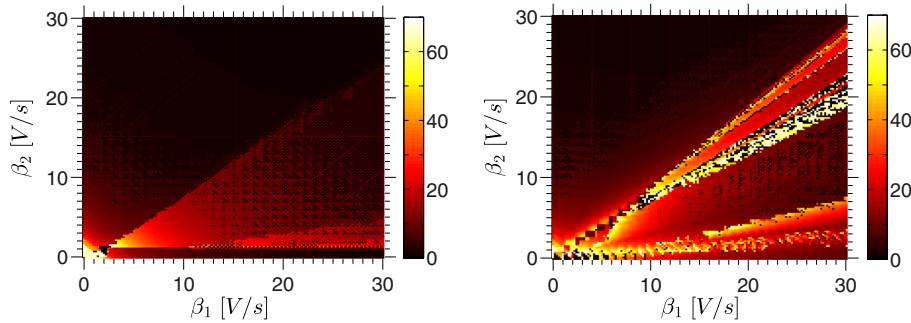

Fig. 5. Synchronization time (colour code) as a function of the coupling strengths β_1 and β_2 (see Fig. 4 for details on the coupling strengths notation and configuration schematics) for the master-slave with intermediary (left panel), and mutual interaction with intermediary (right panel) configurations.

enough (large parameter differences drive the system away from the (1:1) Arnold tongue [5]).

3.2 Three-LCO dynamics

The role of including an intermediary oscillator on the MS (MI) configuration is dealt by analyzing the modification on the synchronization times between the oscillators composing the original MS (MI) configuration. In the following, we note these oscillators as LCO_1 and LCO_3, and the added intermediary as LCO_2. In particular, Fig. 4 shows a scheme of how LCOs 1 and 3 interact (light filled arrows) in MS (left panel) and MI (right panel) configuration, with LCO_2 acting as an intermediary (unfilled arrows). Also, the schematic diagrams show the corresponding notation for the coupling strengths involved in each particular case.

We study the effect of the intermediary by constructing a two dimensional map of the synchronization times as a function of β_1 and β_2, with both coupling strengths varying from 0 to $30\,V/s$ with a step of $0.2\,V/s$. Initial conditions are fixed to $\boldsymbol{V_0} = \frac{V_{cc}}{3}(2,\ 1.75,\ 1.5)$ and the remaining parameters are chosen such that the LCOs are identical ($\lambda = 26\,s^{-1}$ and $\gamma = 1050\,s^{-1}$). Figure 5 shows in colour code such synchronization times in a bidimensional map for the two configurations shown in Fig. 4.
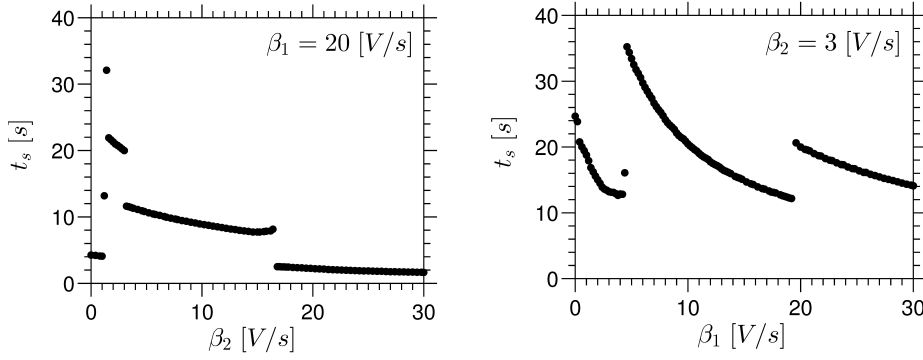
Fig. 6. One-dimensional horizontal (left panel) and vertical (right panel) sections of the map depicted on the left panel of Fig. 5 (i.e., MS with intermediary). Discontinuities in the synchronization times are readily visualized. The section on the left (right) panel is constructed by taking a coupling parameter value of $\beta_2 = 3$ V/s ($\beta_1 = 20$ V/s), with β_1 (β_2) varying from 0 to 30 V/s with a step of 0.2 V/s.

In this figure, for constant values of the β_1, we observe an overall trend in which the synchronization time decreases as β_2 increases. We also observe that the relationship between synchronization times and β_1 is more complex for fixed positive values of $\beta_2 > 0$ than for $\beta_2 = 0$. However, for a wide range of values of β_1, the relationship between the synchronization time and the coupling strength cease to be monotonous and, in fact, the presence of discontinuities is notorious. Surprisingly, for some values of β_1, the synchronization time increases when increasing the coupling strength β_2. The existence of discontinuities in the relationship between the synchronization times and coupling strengths is also notorious. This phenomenon is best illustrated by taking one dimensional sections of the map, as it is shown in Fig. 6.

4 Interpretation of the discontinuities in the synchronization times

To gain a deeper insight in the emergent collective behavior found in the dynamics of 3 identical LCOs, we look at the effect of the master over the slave in the two-LCO dynamics. In general, for an arbitrary phase difference, the effect of the interaction is to speed up the evolution of the slave phase until the phase difference vanishes in the synchronous state. The only exception is when the phase of the slave is slightly advanced in relation with the master in such a way that both oscillators are, for some time interval, simultaneously in their discharging state. In this case, the master delays the slave phase until the synchronous state is reached. The limiting condition between these two cases is given by a critical phase difference $\Delta\phi_c = 2\pi\frac{\lambda}{\lambda+\gamma}$. In our case, as we are dealing with relaxation oscillators (with a fast discharge), this critical phase difference is considerably small $\Delta\phi_c \sim 2\pi/40$. Therefore, if the initial phase difference is less than $\Delta\phi_c$ and the slave phase advanced with respect to the master, the synchronization time is relatively small. On the opposite case, if the initial phase difference is greater than the critical value, the master speeds up the phase of the slave until they are synchronized with a phase difference of 2π. In the latter case, the synchronization time is relatively large. To summarize, in the case of two identical LCOs interacting in a MS configuration, the relationship between the initial phase difference and the synchronization time is discontinuous, and the synchronous state is reached once the phase difference between both oscillators vanishes. In particular, if the phase difference is zero for any time, it will remain zero in the future.

Next, let us consider the dynamics of three LCOs interacting in the MS configuration with an intermediary [Fig. 4(a)], namely, configuration A. In this case, the evolution is far more complex than in the case with only two LCOs. Here, even if the phase difference between the master LCO_1 and the slave LCO_3 is zero for some particular time, the implication of a stable synchronized state between these oscillators is lost. The reason is that the intermediary, LCO_2 is influencing the slave, leading to a phase difference between master and slave.

The synchronization between the master and the slave is only possible after the master is synchronized with the intermediary. In configuration A, the intermediary is unaffected, thus, the master-intermediary interaction is identical to a master-slave configuration in a two LCO system and the synchronization times display similar curves to that of Fig. 3. As mentioned before, when the synchronous state between master and slave is reached, their phase difference will be zero and their discharging times will coincide. Henceforth, when they are synchronized, the influence of LCO_1 and LCO_2 over the slave, LCO_3, can be considered as the interaction of only one identical LCO, $LCO_{1,2}$, with LCO_3 in a MS configuration but with a different coupling strength $\beta_1 + \beta_2$. The initial conditions for this equivalent interaction are the phases at the time in which the master and the intermediary reach their synchronous state. Denoting, the synchronization time between master and intermediary as $t_{s\,1-2}$, the initial conditions is $\phi_{1,2}^i = \phi_2\left(t_{s\,1-2}\right) = \phi_3\left(t_{s\,1-2}\right)$ and $\phi_3^i = \phi_3(t_{s\,1-2})$.

In the light of the preceding analysis, the synchronization time between the master and the slave is expressed as the sum of the synchronization time between the master and the intermediary and the synchronization time between the equivalent $LCO_{1,2}$ and the slave

$$t_{s\,1-3} = t_{s\,1-2} + t_{s\,1,2-3} \tag{5}$$

where $t_{s\,1-3}$ is the synchronization time between master and slave and $t_{s\,1,2-3}$ is the synchronization time between $LCO_{1,2}$ and the LCO_3 in a MS configuration with coupling strength $\beta_1 + \beta_2$ and initial conditions $\phi_{1,2}^i$ and ϕ_3^i. As the dependence of $t_{s\,1-2}$ on the coupling strength is similar to that presented in Fig. 3, the origin of the discontinuities observed in Figs. 5 and 6, lies in the second term of the r.h.s. of Eq. (5). As the quantity $t_{s\,1,2-3}$ depends directly on the initial conditions, $\phi_{1,2}^i$ and ϕ_3^i, it presents a discontinuity provided that the initial phase difference is equal to the critical phase difference $\Delta\phi_c$ previously defined. The specific values of the initial phases, $\phi_{1,2}^i$ and ϕ_3^i, depend on the coupling strength β_1 and β_2 (as they influence $t_{s\,1-2}$), originating the mentioned discontinuities.

In the case of configuration B [Fig. 4(b)], LCO_1 and LCO_3 are in MI configuration. The intermediary destroys the symmetry that occurs with $\beta_2 = 0$. It is necessary again, that LCO_1 and LCO_2 synchronize before for the synchronization between LCO_1 and LCO_3 is possible. Similar arguments to the analysis of the previous configurations lead to the existence of some discontinuities between synchronization times and coupling strengths for the configuration B as well.

Another case of interest is the situation where LCO_2 is not identical to LCO_1 and LCO_3. Such a system is capable of showing other types of synchronization. For autonomous systems (such as three coupled LCO), generalized synchronization has been shown to be possible [19], though synchronization, in general, is not guaranteed. For the MS configuration with intermediate hierarchy, synchronization between LCO_1 and LCO_2 is only possible if the frequency detuning and coupling intensity are such that LCO_2 falls within an Arnold Tongue of the LCO_1-LCO_2 interaction [5]. If it falls within a 1-1 Arnold Tongue, total synchronization is guaranteed. Also, for higher-order Arnold Tongues, LCO_2 will present generalized synchronization with respect to LCO_1, but then the synchronization of LCO_3 with respect to LCO_1 is no

$$\beta_1 = 180 \; [V/s], \; \beta_2 = 12 \; [V/s]$$
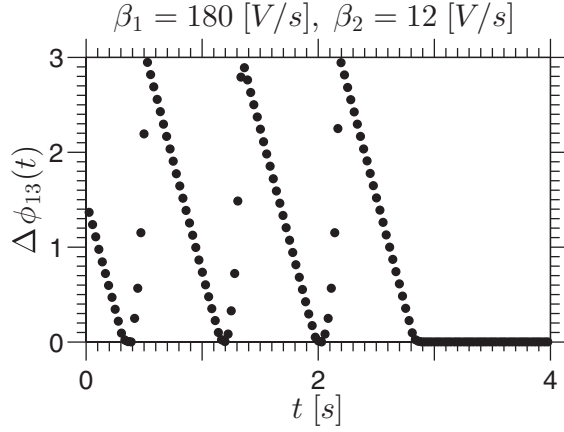

Fig. 7. Phase difference between LCO$_1$ and LCO$_3$ for configuration A shown in Fig. 4, under the same conditions given in Fig. 5. Phase of the LCO$_3$ is first attracted to the phase of LCO 1, but the perturbations made by LCO$_2$ unable the synchronization between them, attracting the phase of LCO$_3$ to the phase of the LCO$_2$, condition that is then perturbed again by LCO$_1$. LCO$_1$ and LCO$_3$ are not ready to synchronize until LCO$_1$ and LCO$_2$ synchronize first, only then the phase difference will evolve to a constant value.
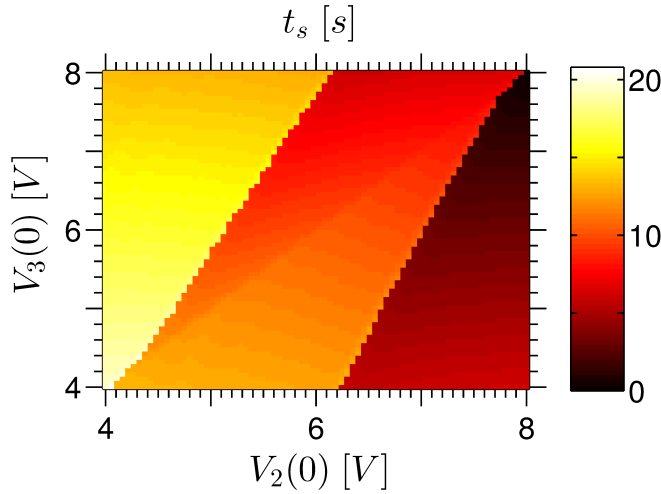
$$t_s \; [s]$$

Fig. 8. Synchronization time as a function of the initial conditions of the LCOs 2 and 3, for fixed coupling strengths. The parameters of the three LCOs are set identical ($\lambda = 26 \, \mathrm{s}^{-1}$ and $\gamma = 1050 \, \mathrm{s}^{-1}$), with coupling strengths $\beta_1 = 4 \, \mathrm{V/s}$ and $\beta_2 = 7 \, \mathrm{V/s}$ and initial condition of LCO 1 is fixed at $V_{01} = 2V_{cc}/3$. Initial conditions of LCOs 2 and 3 are ranged from $V_{cc}/3$ to $2V_{cc}/3$ with a step of $0.053 \, \mathrm{V}$.

longer guaranteed. The case of the MI configuration with non-identical intermediate hierarchy is far more complex and is left for further studies.

Finally, we remark that we are able to derive conditions under which the inclusion of intermediaries reduces the synchronization times between two LCOs. In the case of MS with intermediary, the necessary condition to reduce the synchronization time with respect to the two-LCO configuration is

$$t_{s1-2} + \max \left\{ t_{s1,2-3} \right\}_{\Delta\phi} < t_{s1-3(free)}, \tag{6}$$

where $t_{s1,2-3}$ is the synchronization time between $LCO_{1,2}$ and LCO_3, and $t_{s1-3(free)}$ is the synchronization time between LCO_1 and LCO_3 when LCO_2 is absent. The second term on the left in Eq. (6) is the maximum value of $t_{s12,3}$ among every possible pair of initial conditions $\phi_{1,2}^i$ and ϕ_3^i.

5 Conclusions

Groups of two and three coupled LCOs present a number of interesting synchronization properties. This work includes results for identical oscillators, though further studies indicate the validity of our results and analysis for groups of quasi-identical oscillators and, in smaller regions, for larger frequency mismatches between the LCOs. Specifically, we studied the influence of coupling strengths over synchronization times for groups of two LCOs in MS and MI configurations, and for groups of three LCOs in a MS configuration with an intermediate hierarchy and a MI configuration with an intermediate hierarchy. When comparing synchronization times for two-LCO and three-LCO dynamics, we found that unlike systems of two LCOs, synchronization times for systems of three LCOs do not always decrease when coupling strengths increases, and larger non-synchronous regions are found. In three LCO dynamics, we found an interesting, unexpected effect: a discontinuous relationship between synchronization times and coupling strengths or initial conditions. Furthermore, we presented a qualitative analysis of the dynamics of three LCOs coupled in configurations (a) and (b) of Fig. 4, that justifies the existence of such discontinuities. In the light of this analysis, we found sufficient conditions for intermediate hierarchy to reduce synchronization times with respect to the analogue configurations without the intermediate hierarchy. These findings could be relevant for the construction and interpretation of functional networks, such as those constructed from brain or climate data [20,21].

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