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Detecting and quantifying temporal correlations in stochastic resonance via information theory measures

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Abstract. We show that Information Theory quantifiers are suitable tools for detecting and for quantifying noise-induced temporal correlations in stochastic resonance phenomena. We use the Bandt & Pompe (BP) method [Phys. Rev. Lett. **88**, 174102 (2002)] to define a probability distribution, P, that fully characterizes temporal correlations. The BP method is based on a comparison of neighboring values, and here is applied to the temporal sequence of residence-time intervals generated by the paradigmatic model of a Brownian particle in a sinusoidally modulated bistable potential. The probability distribution P generated via the BP method has associated a normalized Shannon entropy, $\mathcal{H}[P]$, and a statistical complexity measure, $\mathcal{C}[P]$, which is defined as proposed by Rosso et al. [Phys. Rev. Lett. **99**, 154102 (2007)]. The statistical complexity quantifies not only randomness but also the presence of correlational structures, the two extreme circumstances of maximum knowledge ("perfect order") and maximum ignorance ("complete randomness") being regarded an "trivial", and in consequence, having complexity $\mathcal{C} = 0$. We show that both, $\mathcal{H}$ and $\mathcal{C}$, display resonant features as a function of the noise intensity, i.e., for an optimal level of noise the entropy displays a minimum and the complexity, a maximum. This resonant behavior indicates noise-enhanced temporal correlations in the sequence of residence-time intervals. The methodology proposed here has great potential for the precise detection of subtle signatures of noise-induced temporal correlations in real-world complex signals.

PACS. 05.40.-a Fluctuation phenomena, random processes, noise, and Brownian motion – 05.40.Ca Noise – 05.45.Tp Time series analysis – 02.50.-r Probability theory, stochastic processes, and statistics

1 Introduction

Stochastic resonance [1,2], originally proposed to explain the occurrence of ice ages [3–5], is nowadays a well-known phenomenon, has been observed in a wide variety of systems [6–9] and is of major relevance in many areas of science, being used, e.g., by many biological sensing organs that take advantage of background noise [10–14].

A lot of effort has been devoted to the precise detection and quantification of stochastic resonance. Traditionally, stochastic resonance has been quantified in terms of the intensity of a peak in the power spectrum, via the computation of the signal-to-noise ratio (SNR) from power spectra taken at fixed amplitude and frequency of the applied external signal and different noise levels; the resulting plot displaying a maximum of the SNR at a certain noise level [2]. Another well established indicator is the intensity of the peaks in the residence-time distribution function [1,15]. This method involves the calculation of the strength of the nth peak, P_n, of the residence times distribution, which also displays a maximum at a certain level of noise. However, the noise levels that maximize SNR and the different peaks of the residence times distribution are not necessarily the same. Moreover, when quantifying stochastic resonance using the strengths of the peaks of the distribution, the phenomenon is seen as a synchronization effect occurring whenever the stochastic time scale (Kramer's switching time) matches the time scale of the applied signal, a condition that can be realized by tuning either the noise intensity or the frequency of the applied signal [8,16].

It would be interesting to characterize the response of nonlinear systems to noise in terms of more subtle

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measures than those based on the power spectra and the distribution of the switching intervals. To quantify aperiodic stochastic resonance, in which the response of a nonlinear system to a subthreshold information-bearing signal is optimized by the presence of noise; in [17] a "trans-information" measure was used to show that noise optimizes the rate of information transfer; in [18] the information transmitted by the output spike train of an integrate-fire model neuron was quantified in terms of Shannon mutual information between the transmitted and received signal; in [19], dynamical conditional and Kullback entropies were employed to relate stochastic resonance to the synchronization of switching events induced by the applied periodic signal; in [20], information-theoretic measures were employed to asses the role of measurement (detector) noise, and the issue of whether or not one can improve a detector performance by adding noise.

Measures based on information theory have also been proposed for quantifying stochastic resonance in ion channels. A measure quantifying conventional SR (with a periodic input signal of infinite duration), aperiodic SR (with an aperiodic input signal of infinite duration), and non-stationary SR (with an arbitrary signal of finite duration) was introduced in [21]. Conductance fluctuations, which switch at random times between two values, were analyzed via the τ information, the mutual information, and the rate of information gain. An analytical formula for the rate of information gain was derived, which was applied to the study SR in a potassium channel. For periodic SR, in [22] the rate of information gain was compared to the signal-to-noise ratio and a non-equivalence between the two measures was established, despite their apparent similarity in the limit of weak signals. Nonstationary SR was studied by Goychuk and Hänggi employing the well-known Kullback-Leibler information distance (see [23] in this special issue). In particular the authors studied how to quantify information transfer in the situation of non-stationary information-carrying input signals of finite duration.

Stochastic resonance phenomena can also involve noise-induced *temporal* order. Several Information Theory measures have been proposed to quantify temporal correlations in a system [24–26], and to the best of our knowledge, they have been overlooked for detecting the ordering role of noise in stochastic resonance phenomena. Measures of complexity defined for symbol sequences have been used in [28] to analyze the dynamics of a stochastic bistable system, both, with additive white noise (Kramers problem) and when the barrier fluctuates due to additional external colored noise. It was shown that the phenomenon of resonant activation occurs accompanied with extreme values of most complexity measures.

We have recently shown [29] that the normalized Shannon entropy, $\mathcal{H}[P]$, and the Statistical Complexity Measure, $\mathcal{C}[P]$, defined as proposed by Rosso and coworkers [30], are suitable tools for detecting subtle signatures of noise-induced temporal order in a bistable system, when the barrier is periodically modulated with a subthreshold external signal. The method involves two key ingredients: (i) an appropriate reconstruction of the attractor from a given time series (i.e., defining a suitable embedding phase space), and (ii) defining an appropriate partition of the phase space, that results in ordinal patterns having a probability distribution, P, that fully characterizes the temporal correlations in the system. Once the distribution P is defined, the calculation of the normalized Shannon entropy, $\mathcal{H}[P]$, and the Statistical Complexity Measure, $\mathcal{C}[P]$, are straight forward. We demonstrated that both Information Theory quantifiers display resonant features as a function of the noise intensity, i.e., for an optimal level of noise the entropy displays a minimum and the complexity, a maximum. We applied the method to time series generated by two well-known models: a Brownian particle in a sinusoidally modulated bistable potential, and the FitzHugh-Nagumo model of excitable systems, for parameters such that the model displays coherence resonance. In the first case, we analyzed the sequence of residence-time intervals; in the second case, the sequence of inter-spike intervals. Here we focus only on stochastic resonance phenomena and describe in detail the methodology used for detecting noise-induced temporal correlations in the sequence of residence-time intervals.

2 Information theory quantifiers: normalized Shannon entropy and MPR-statistical complexity

An information measure, $\mathcal{I}[P]$, is typically a quantity that characterizes a given probability distribution, $P = \{p_j;\ j = 1, \ldots, N\}$. Shannon's entropy [31],

$$\mathcal{S}[P] = -\sum_{j=1}^{N} p_j \ln(p_j), \tag{1}$$

is regarded as the classical information measure of the uncertainty associated to the physical processes described by the probability distribution P. If $\mathcal{S}[P] = 0$ we are in a position to predict with certainty *which* of the possible outcomes j will actually take place. Our *knowledge* of the process described by the probability distribution P is maximal. On the other hand, our *ignorance* is maximal for the uniform distribution, $P_e = \{p_j = 1/N;\ j = 1, \ldots, N\}$. This two extreme circumstances of (i) maximum foreknowledge ("perfect order") and (ii) maximum ignorance (or "complete randomness") can be regarded as trivial. Certainly, they possess no structure to speak of.

Complexity denotes a state of affairs that one can easily appreciate when confronted with it; however, is rather difficult to define it quantitatively (probably due to the fact that there is no universal definition of complexity). In between the two special instances of perfect order and complete randomness, a wide range of possible degrees of physical structure exists that should be reflected in the features of the underlying probability distribution, $P \equiv \{p_j\}$. One would like that the degree of correlational structures would be adequately captured by some functional $\mathcal{C}[P]$ in the same way that Shannon's entropy [31] "captures" randomness.

In trying to ascertain what is meant by statistical complexity, different measures have been proposed in the literature [24,32–37]. One can begin by excluding processes that are certainly not complex, such as those exhibiting periodic motion. Additionally, a purely random process cannot be considered complex either, notwithstanding its irregular and unpredictable character, because it does not contain any non-trivial, patterned structure. Statistical complexity is often characterized by the paradoxical situation of a complicated dynamics generated from relatively simple systems. Obviously, if the system itself is already involved enough and is constituted by many different parts, it clearly may support a rather intricate dynamics, but perhaps without the emergence of typical characteristic patterns [38]. Therefore, a complex system does not necessarily generate a complex output. Statistical complexity has then to do with patterned structures hidden in the dynamics, emerging from a system which itself can be much simpler than the dynamics it generates [38].

Assessing the degree of unpredictability and randomness of a system is not automatically tantamount to adequately grasping the correlational structures that may be present, i.e. to be in a position to capture the relationship between the components of a physical system [34,36]. These correlational structures influence, of course, the functional form of the probability distribution that is able to describe the physics one is interested in. In other words, randomness, on the one hand, and structural correlations on the other one, are not independent aspects of the dynamics.

A suitable candidate to "capture" complexity is an information measure that has come to be called the statistical complexity. This complexity, $\mathcal{C}[P]$, vanishes in the two special extreme instances mentioned above (perfect order and complete randomness).

For a given probability distribution, P, the amount of "disorder" is defined as [39]

$$\mathcal{H}[P] = \mathcal{I}[P]/\mathcal{I}_{max}, \tag{2}$$

where $\mathcal{I}_{max} = \mathcal{I}[P_{max}]$ and P_{max} is the probability distribution which maximizes the information measure. If $\mathcal{I}$ is Shannon's entropy, P_{max} is the uniform distribution, P_e, and the disorder is then the normalized Shannon entropy, $\mathcal{H}_S$, with $0 \leq \mathcal{H}_S \leq 1$.

It follows that a reasonable definition of the statistical complexity measure must adopt some kind of distance $\mathbf{D}$ to the uniform distribution P_e [24–26]. For such a purpose the "disequilibrium", $\mathcal{Q}$, is defined as

$$\mathcal{Q}[P, P_e] = Q_0 \cdot \mathbf{D}[P, P_e], \tag{3}$$

where Q_0 is a normalization constant such that $0 \leq \mathcal{Q} \leq 1$. The disequilibrium $\mathcal{Q}$ reflects on the system's "architecture", being different from zero if there are privileged, or more likely states among the accessible ones. Consequently, we will adopt the following functional form, introduced originally by López-Ruiz et al. (LMC) [24], for the statistical complexity measure:

$$\mathcal{C}[P] = \mathcal{H}_S[P] \cdot \mathcal{Q}[P, P_e]. \tag{4}$$

This quantity reflects on the interplay between the amount of information stored in the system and its disequilibrium.

In the LMC definition of complexity, the disequilibrium, $\mathcal{Q}[P, P_e]$, is computed using the Euclidean distance from P to the uniform distribution P_e. Martín, Plastino, and Rosso [25] proposed a modification of the LMC-complexity consisting of the use of Wootters statistical distance [40] instead of the Euclidean distance. An alternative definition of the disequilibrium was proposed by Lamberti et al. [26] using the Jensen-Shannon divergence, which is the symmetric form of the Kullback-Leiber relative entropy, $K[P_1|P_2]$ [27] (i.e., $J_S[P_1, P_2] = (K[P_1|P_2] + K[P_2|P_1])/2$).

The statistical complexity that computes the disequilibrium using the Jensen-Shannon divergence (in the following, referred to as MPR-complexity) is the one used in this work and it was recently shown [30] that is suitable for distinguishing time series generated by stochastic and by chaotic systems, and able to classify different degrees of stochasticity. Moreover, the MPR-complexity was shown to be suitable for distinguishing Gaussian from non-Gaussian process, and for distinguishing among different degrees of correlations (colored noises).

Summarizing the above discussion, the MPR statistical complexity for a probability distribution $P = \{p_j; j = 1, \ldots, N\}$, with N being the number of possible states of the system, can be cast as

$$\mathcal{C}[P] = \mathcal{H}_S[P] \cdot \mathcal{Q}_J[P, P_e], \tag{5}$$

in which the disorder is measured by the normalized Shannon entropy and the disequilibrium $\mathcal{Q}_J$ is defined in terms of the Jensen–Shannon divergence [26]:

$$\mathcal{Q}_J[P, P_e] = Q_0 \cdot \mathcal{J}_S[P, P_e], \tag{6}$$

where

$$\mathcal{J}_S[P_1, P_2] = \{S[(P_1 + P_2)/2] - S[P_1]/2 - S[P_2]/2\} \tag{7}$$

and Q_0 is a normalization constant, equal to the inverse of maximum possible value of $\mathcal{J}_S[P, P_e]$,

$$Q_0 = -2\left\{\left(\frac{N+1}{N}\right)\ln(N+1) - 2\ln(2N) + \ln N\right\}^{-1}. \tag{8}$$

We stress the fact that the above defined statistical complexity is the product of two normalized entropies (the Shannon entropy and Jensen-Shannon divergence), but is a nontrivial function of the entropy, because it depends on two different probabilities distributions, the one corresponding to the state of the system, P, and the uniform distribution, P_e. It was shown in [41] that for a given value of $\mathcal{H}$, there is a range of possible values for $\mathcal{C}$. Thus, evaluating the complexity provides additional insight in the details of a system's probability distribution, which are not described by the entropy [30,41]. In order to study the time evolution of the statistical complexity, a diagram of $\mathcal{C}$ versus $\mathcal{H}$ can be used, the $\mathcal{H} \times \mathcal{C}$ plane (in this case, $\mathcal{H}$ can be regarded as an arrow of time [42]). Also, this kind of diagram has been used to study changes in the dynamics of a system originated by modifications of some characteristic parameters [24,26,43–46].

3 The Bandt and Pompe probability distribution

Let's consider a system whose output consists of a time series $\{x_s : s = 1, \ldots, M\}$. A critical point for estimating the complexity of this system is finding a suitable probability distribution, P, that fully characterizes the time series under study.

This requires a partition of a D-dimensional embedding space that will, hopefully, reveal the ordinal-structure of the time series. Here, for defining this partition and its associated probability distribution, we use the method proposed by Bandt and Pompe [47], that is based on a comparison of neighboring values. The Bandt and Pompe P-generating algorithm requires that the system fulfils a weak stationary condition (for $k < D$, the probability for $x_s < x_{s+k}$ should not depend on s) and that enough data is available for a correct reconstruction of the attractor.

Given an embedding dimension $D > 1$, one is interested in "ordinal patterns" of order D generated by assigning to each time s a D-dimensional vector of values pertaining to the previous times:

$$(s) \mapsto \left(x_{s-(D-1)}, x_{s-(D-2)}, \ldots, x_{s-1}, x_s\right). \qquad (9)$$

By the "ordinal pattern" related to the time (s) we mean the permutation $\pi = (r_0, r_1, \ldots, r_{D-1})$ of the symbols $(0, 1, \ldots, D-1)$ defined by

$$x_{s-r_{D-1}} \leq x_{s-r_{D-2}} \leq \ldots \leq x_{s-r_1} \leq x_{s-r_0}. \qquad (10)$$

In order to obtain a unique result, we consider that $r_i < r_{i-1}$ if $x_{s-r_i} = x_{s-r_{i-1}}$.

As an example, with an embedding dimension $D = 3$ the time series

$$\{x_i\} = \{3.4, 5.1, 6.8, 11.5, 1.1, 2.3, \ldots\} \qquad (11)$$

has associated the following sequence of "ordinal patterns", each composed by three symbols $(0, 1, 2)$,

$$\begin{aligned}
(3.4,\ 5.1,\ 6.8) &\mapsto (0\ 1\ 2)\\
(5.1,\ 6.8,\ 11.5) &\mapsto (0\ 1\ 2)\\
(6.8,\ 11.5,\ 1.1) &\mapsto (2\ 0\ 1)\\
(11.5,\ 1.1,\ 2.3) &\mapsto (1\ 2\ 0)\\
&\ldots
\end{aligned}$$

In an "ordinal pattern" the symbol sequence comes from a comparison of neighboring values: for example, for the vector $(v_0, v_1, v_2) = (6.8,\ 11.5,\ 1.1)$, we have $v_2 < v_0 < v_1$, which results in the "ordinal pattern" (2 0 1).

For all the $D!$ possible permutations π of order D, the probability distribution $P = \{p(\pi)\}$ of "ordinal patterns" is defined as

$$p(\pi) = \sharp\{s | s \leq \mathcal{Y}; (s)\ \texttt{has type}\ \pi\}/\mathcal{Y}, \qquad (12)$$

where $\mathcal{Y} = M - D + 1$ and $\sharp$ stands for "number".

The method proposed by Bandt and Pompe [47] for evaluating the probability distribution P is based on the details of the attractor-reconstruction procedure. Bandt and Pompe consider a partition of the D-dimensional state space determined by the intersections of $D!$ hyper-planes

$$\begin{aligned}
\Re^D : x_1 = x_2, x_1 = x_3, \ldots, x_1 = x_D;\\
x_2 = x_3, \ldots, x_2 = x_D;\\
\ldots;\\
x_{D-1} = x_D. \qquad (13)
\end{aligned}$$

Each permutation π of order D can be associated with one of the connected pieces determined by the partition. In other words, an "ordinal pattern" represents one connected piece of $\Re^D$, and the union of all pieces is the total state space $\Re^D$. The probability distribution P of "ordinal patterns" is given by the frequency, in the attractor structure, of each piece (pattern). P is assigned by "counting" the times that the attractor visits each piece (see Eq. (12)). In particular, if the attractor is symmetric with respect to the hyperplanes, all the connected pieces have the same frequency and thus the distribution of ordinal patterns is uniform: the attractor "visits" all the partition pieces with the same frequency. Consequently, the information provided by the time series so as to predict geometric locations of successive D-strings vanishes and the entropy is maximal ($S_{max} = \ln D!$ and $\mathcal{H}_S = 1$). On the other hand, if the situation is such that the attractor remains always within just one of the connected pieces, one can "predict" with certainty $\mathcal{H}_S = 0$.

The advantages of Bandt and Pompe method reside in (a) its simplicity, (b) the fast calculation process [48], (c) its robustness, and (d) its invariance with respect to nonlinear monotonous transformations. The Bandt and Pompe's methodology can be applied to any type of time series (regular, chaotic, noisy, or reality based), with a weak stationary assumption [47].

It is also important to remark that for the applicability of Bandt and Pompe's technique we need not to assume that the time series under analysis is representative of a low dimensional dynamical system. Because the probability distribution, P, constructed in this way takes into account the temporal structure of the time series, not only the geometrical structure of the reconstructed attractor, but also *causal information*, is incorporated in the partition process that yields $P \in \Omega$ (with Ω the probability space) [49].

4 Results

In this section use Information Theory quantifies (the normalized Shannon entropy and the MPR-statistical complexity) to analyze time series generated by a Brownian particle in a sinusoidally modulated bistable potential. The data, $\{x_1, x_2, \ldots\}$, are consecutive residence-time intervals (RTIs) generated from numerical simulations of a bistable system, subjected to noise, and to an external subthreshold periodic signal (i.e., a small-amplitude signal that, by itself, does not induce switchings):

$$\frac{dx}{dt} = -\frac{\partial V(x,t)}{\partial x} + \xi(t) = x - x^3 + A\sin(\omega t) + \xi(t). \quad (14)$$

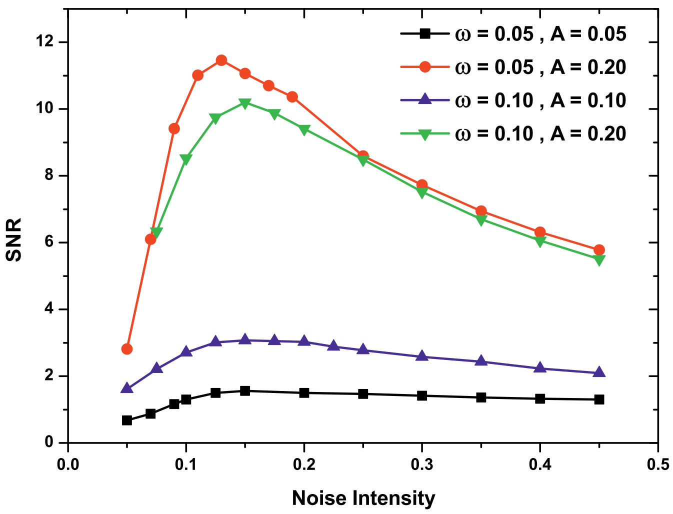

Fig. 1. (Color online) Quantifying stochastic resonance for varying noise intensity and various values of the amplitude and the modulation frequency. The standard indicator (the signal-to-noise ratio in arbitrary units) is plotted vs. the noise intensity, $\mathcal{D}$.

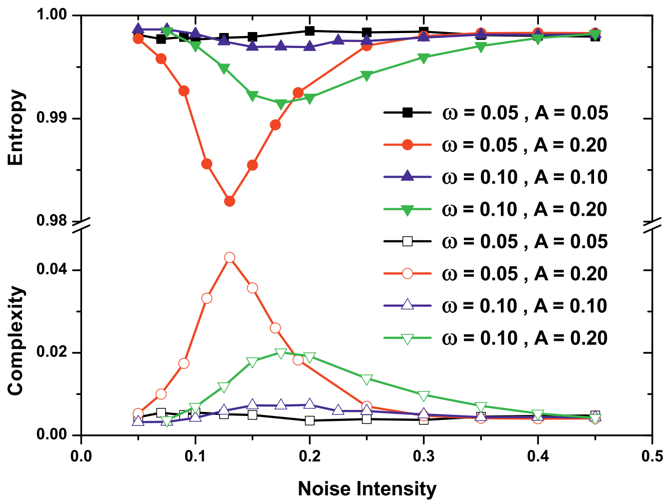

Fig. 2. (Color online) Information Theory quantifiers: the MPR statistical complexity, $\mathcal{C}$, (empty symbols) and the normalized Shannon entropy, $\mathcal{H}$, (filled symbols) both in arbitrary units are plotted vs. the noise intensity, $\mathcal{D}$. Parameters are as in Figure 1.

Here $V(x,t) = -x^2/2 + x^4/4 - Ax\sin(\omega t)$, A is the amplitude and ω is the frequency of the external signal and ξ is a Gaussian noise having zero mean, and correlation function $\langle \xi(t)\xi(t')\rangle = 2\mathcal{D}\delta(t-t')$, with $\mathcal{D}$ being the noise intensity.

We compute the normalized Shannon entropy and the MPR-statistical complexity from a sequence of M consecutive residence times intervals: if the switchings occur at times $\{t_0, t_1, t_2 \dots\}$ then $\{x_s = t_s - t_{s-1} : s = 1, \dots, M\}$. The method consists of three basic steps, as explained in the previous sections:

(1) We apply the Bandt and Pompe method to associate to the time series of RTIs $\{x_s\}$ a sequence of "ordinal patterns" of order D, composed by the symbols $(0, 1, \dots, D-1)$.
(2) We then calculate the probability distribution of "ordinal patterns", $P\{p(\pi)\}$.
(3) We compute the normalized Shannon entropy, $\mathcal{H}$, and the MPR-statistical complexity, $\mathcal{C}$, associated to the probability distribution $P\{p(\pi)\}$.

The selection of a suitable the embedding dimension, D, is important for obtaining a meaningful probability distribution, because D determines not only the number of accessible states (equal to $D!$), but also, the length of the time series, M, needed to have a reliable statistics (the condition $M \gg D!$ must be satisfied). For practical purposes Bandt and Pompe use $3 \leq D \leq 7$ [47]; here, unless otherwise stated, we use $D = 6$ and analyze time series of $M = 60\,000$ consecutive residence times intervals.

Let us first present the well-known quantification of stochastic resonance in terms of the signal-to-noise ratio (SNR), which exhibits a maximum for an optimal noise level, for various values of A and ω, as shown in Figure 1. Consider now the Information Theory based quantifiers: $\mathcal{C}$ and $\mathcal{H}$. It can be seen in Figure 2 that resonant-like behavior is also observed: the shape of the curves obtained is consistent with the behavior of the conventional quantifier, the SNR. For optimal noise levels we notice extreme values: maximum complexity and minimum entropy. We have previously shown that resonant-like behavior occurs when varying either the noise intensity, $\mathcal{D}$, or the external modulation frequency, ω [29]. Moreover, we have shown that the resonant-like behavior is associated with noise-enhanced temporal correlations: when we analyzed the original series of residence times intervals, but re-arranged the data randomly, the resonant-like behavior disappeared, and we obtained values $\mathcal{H} \sim 1$ and $\mathcal{C} \sim 0$, which are compatible with a pure random process, confirming in this way that the resonant-like structure observed in the two Information Theory quantifiers computed from the original series can be attributed to noise-induced temporal correlations in the sequence of switchings.

The results in Figure 2 seem to suggest that the Shannon entropy and MPR-complexity measure yield the same information; however, as explained before, $\mathcal{C}$ is a nontrivial function of $\mathcal{H}$, and for each value of $\mathcal{H}$ there is a range of possible $\mathcal{C}$-values: $\mathcal{C}_{min} \leq \mathcal{C} \leq \mathcal{C}_{max}$ (the procedure for calculating $\mathcal{C}_{min}$ and $\mathcal{C}_{max}$ is presented in [41]). By plotting the results in the $\mathcal{C}$-$\mathcal{H}$ plane, Figure 3, we obtain a close loop curve (that in this case is almost a line), with the left extreme being the resonant-like situation: maximum complexity, minimum entropy.

We verified that the results are robust with respect to the value of D: similar results were found with $D = 4$ and 5, as can be seen in Figure 4. Also, the results are robust when the total length of the time series is reduced, as can be seen in Figure 5. This indicates that the probability distribution of the ordinal patters is stationary, validating in this way the requirement $M \gg D!$.

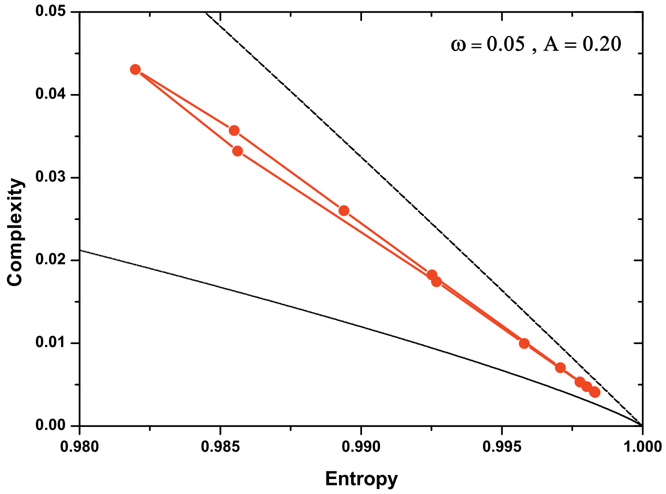

Fig. 3. (Color online) $\mathcal{C}$ vs. $\mathcal{H}$. The solid lines indicate the boundary values, $\mathcal{C}_{max}$ and $\mathcal{C}_{min}$.

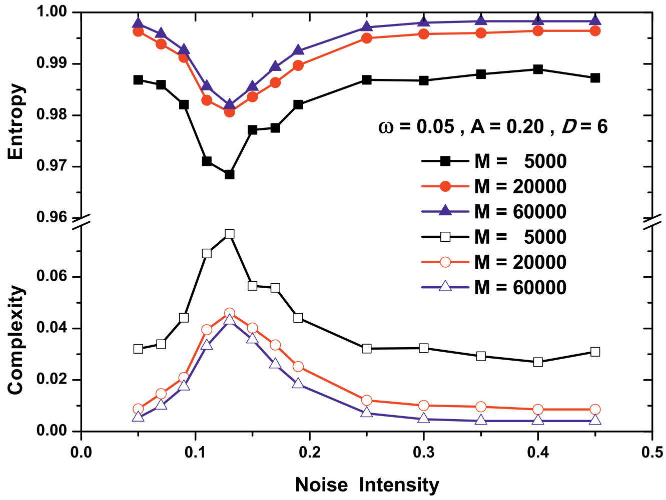

Fig. 5. (Color online) $\mathcal{C}$ and $\mathcal{H}$ vs. the noise intensity for various lengths of the time series, M.

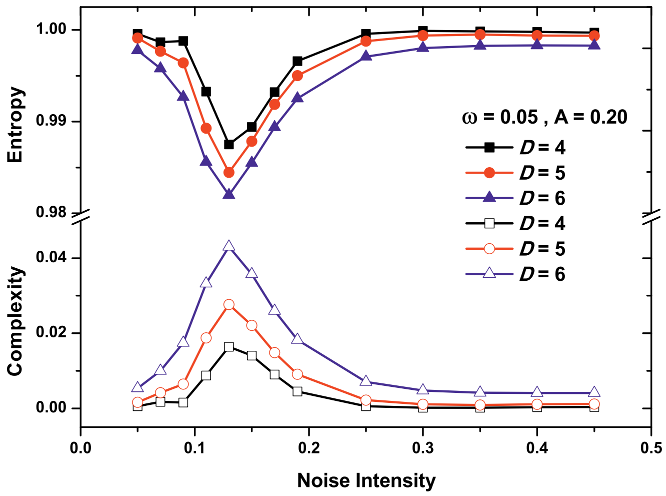

Fig. 4. (Color online) $\mathcal{C}$ and $\mathcal{H}$ vs. the noise intensity for various embedding dimensions.

5 Discussion and conclusions

We interpret the above results in the following terms: since the data under investigation consist of time intervals between noise-induced switchings (residence time intervals, RTIs), one can expect that the two quantifiers are $\mathcal{H} \approx 1$ and $\mathcal{C} \approx 0$, corresponding to probability distribution of "ordinal patterns", $P\{p(\pi)\}$, very near to the uniform distribution (all ordinal patterns being equally probable), appearing with the same probability in the time series. However, at certain noise levels there is an enhancement of time-correlations, resulting in the probability distribution of the ordinal patterns, $P\{p(\pi)\}$, being different from the uniform distribution. Correspondingly, there is a decrease of the entropy (reveling some degree of order), and also, an increase of the statistical complexity measure. We identified this behavior as the hallmark of resonant-like behavior.

Figure 4 demonstrates that the resonant-like behavior becomes more pronounced when increasing the embedding dimension, D. Increasing D means that we are analyzing longer "ordinal patterns", and therefore, it suggest that the Information Theory quantifiers are detecting a kind of long-range noise-induced order. Further studies are in process to clarify this point.

In summary, the Bandt and Pompe methodology for the determination of the probability distribution associated with the time series of residence times intervals (RTIs) allows to replace the original continuous-time, continuous-state stochastic process by a discrete-time, discrete-state process. The information about the precise timing of the switchings and about the duration of the residence time intervals is lost, which might be considered as a drawback of the method as it takes into account only "gross-correlations", fine details being omitted. The precise timing of the switchings are important for information transfer; however, an advantage of the Bandt and Pompe method is that it removes the influence of the value of the discretization time bin, which is a problem for determining the information entropy of renewal processes in absolute terms (see, e.g. [21]). It will be interesting in a future work to characterize the time-ordering role of noise by using the Kullback entropy, which takes into account the precise timing of the switchings and the change of the absolute length of RTIs. This measure has the advantage of defining the entropy change (gained information) in absolute terms, not suffering from the time bin problem, and at the same time, not losing any relevant information.

We have shown that the Information Theory measures, the normalized Shannon entropy and MPR statistical complexity, can be employed to detect and quantify resonant-like behavior in the form of enhanced temporal order, induced by the variation of a system parameter or by the variation of the noise level. The success of the method is based on an appropriate reconstruction of the attractor, and on an appropriate partition of the phase space, that results in ordinal patterns having a probability distribution that fully characterizes the temporal correlations in the system.

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