Transient low-frequency fluctuations in semiconductor lasers with optical feedback

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(Received 26 November 2009; published 11 March 2010)

Time-delayed systems often exhibit multistability of coexisting attractors, which can result in long chaotic transients on the way to one of the coexisting states. Strong enough noise can transform this transient chaos into noise-sustained dynamics. Here we study the interplay between delay-induced multistability, chaotic transients, and noise, in the case of a semiconductor laser with optical feedback from an external reflector. The time-delayed feedback renders the laser multistable, with a set of coexisting fixed points, and induces dynamical events called low-frequency fluctuations (LFFs), consisting of sudden intensity dropouts at irregular times. The deterministic Lang-Kobayashi model shows that, for a large range of realistic laser parameters, the LFFs are just a transient dynamics toward a stable fixed point. Here we analyze the statistical properties of the transient LFF dynamics and investigate the influence of various parameters. We find that realistic values of the noise strength do not affect the average transient time or its distribution, provided the model includes an explicit delay. On the other hand, nonlinear gain saturation has a strong effect: it increases both the duration of the LFF transients and the probability of noise-induced escapes from the stable fixed point. Our results suggest that the LFFs observed experimentally can be, at least in part, sustained by the interplay of noise and various nonlinear effects, which are phenomenologically represented by a gain saturation coefficient.

DOI: 10.1103/PhysRevA.81.033820 PACS number(s): 42.65.Sf, 42.55.Px, 05.45.−a, 42.60.Mi

I. INTRODUCTION

Certain nonlinear dynamical systems exhibit transient chaotic trajectories while on their way to a stable (usually nonchaotic) attractor [1]. These chaotic transients result from the existence of nonattracting chaotic invariant sets in the phase space of the system [2]. In low-dimensional dynamical systems, the nonattracting chaotic sets are necessarily low dimensional, and the average duration (and in general the statistics) of chaotic transients can be expected to be affected by noise and by the system parameters. On the other hand, when the nonattracting chaotic set is high dimensional (what happens, for instance, in systems with delay), it is not clear whether the sensitivity to noise and system parameters is maintained, or whether the set dimension takes over and dominates the transient dynamics of the system completely. In this article we address this question in a delayed system of special technological importance, namely in a semiconductor laser with delayed feedback.

In many applications of semiconductor lasers, the device is subject to optical feedback from an external reflector. This feedback can either optimize the laser performance (by lowering its threshold and reducing the intensity noise) or it can be detrimental, inducing multistability and chaos [3,4]. A well-known optical feedback-induced instability is the so-called low-frequency-fluctuation (LFF) regime, which occurs when the laser operates close to its solitary threshold and is affected by a moderately strong optical feedback. In this regime, the laser intensity displays erratic dropouts in the emitted light intensity, which are actually the envelope of fast picosecond pulses [5,6], followed by a gradual recovery process. Many studies in the literature have tried to address the question of whether the initiation of LFF dropouts and the subsequent recovery are deterministic or stochastic processes.

A well-known feature of the LFF dynamics is that, as the bias current is increased, the average time interval between consecutive dropouts decreases and the dropouts become increasingly frequent and begin to merge [7]. Through this gradual transition, the laser output becomes increasingly irregular with increasing bias current. For large enough bias current no dropouts are observed but rather a completely irregular intensity time trace, a regime which has been termed fully developed coherence collapse. Another feature of the LFF regime is that, in a wide range of parameters, it coexists with stable emission [8], with the relative duration of the stable emission state (in comparison with the duration of the LFF state) depending strongly on the bias current, the feedback strength, and the phase-amplitude coupling factor (α factor) [9,10]. The LFF dynamics also presents very interesting characteristics from the stochastic point of view, because it renders the laser an excitable system, and thus provides a controllable setup to investigate noise-induced phenomena such as coherence resonance [11–13] and stochastic resonance [14,15].

A well-known and widely employed model to study the LFF dynamics is the Lang-Kobayashi (LK) model [16], which assumes as main simplifications single-mode emission and a single reflection in the external cavity. The LK model is a set of delay-differential equations with a rich variety of dynamics. Depending on the feedback parameters there is multistability of steady state (fixed-point) solutions, which are known as external cavity modes (ECMs). These fixed points are stable focus, saddle points, or unstable focus, the saddle points usually being referred to as antimodes [17]. Within

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the framework of the LK model, the LFF dynamics has been interpreted as due to chaotic itinerancy with a drift [18], an effect also known as the "Sisyphus effect" [19]: the trajectory tends to move toward the ECM with the maximum gain, which is a stable focus; however, this mode is located in the phase space in a region close to the antimodes, and a "collision" with one of them expels the trajectory toward the solitary laser steady state, triggering an intensity dropout. The resulting chaotic dynamics can be strongly high dimensional for realistic parameters [20]. Alternatively, experimental data on the statistics of the time intervals between power dropouts are consistent with the assumption that noise triggers the dropouts and that after a dropout the laser recovers in a deterministic fashion [21,22].

Despite the fact that the LFF dynamics has been extensively studied within the framework of the LK model, few studies have addressed the issue that this dynamics is actually *a transient* regime for a wide range of realistic laser parameters [23–25]: simulations of the noise-free LK model show that the power dropouts vanish once the trajectory reaches the basin of attraction of a stable ECM. At that point, ejecting the trajectory away from the stable ECM (so the LFF can restart) requires the presence of a certain amount of noise. A key parameter that increases the duration of the LFF transient is the phase-amplitude coupling factor [25]. When a value of $\alpha \sim$ 5–6 is used, simulations of the LK model predict stationary LFFs (in the sense that the dropouts last for time intervals longer than 10–100 ms); however, if a value of $\alpha \sim 2.5$–3.5 is used, which is realistic for currently available diode lasers, the simulations predict that the dropouts will die out after 1–10 ms. This is in good agreement with earlier observations by Heil *et al.* [10], who found that stable emission was more probable when decreasing the linewidth enhancement factor, so that for $\alpha \sim 1$ robust stable emission was observed over a very wide parameter range.

The aim of this article is to characterize the statistical features of the transient LFF dropouts and the influence on these statistics of the high dimensionality of the underlying nonattracting chaotic set. With this aim, we simulate the laser turn on by choosing *random* initial conditions in the vicinity of the solitary laser steady state and integrate numerically the resulting trajectory. The LFF lifetime, T_{LFF}, is defined as the time taken by the intensity fluctuations to decrease below a chosen threshold, which occurs when the trajectory falls into the basin of attraction of one of the stable ECMs. We find that typical noise levels do not *significantly* affect the average transient time nor its probability distribution function (PDF): both the deterministic model, with no noise source included in the rate equations, and the stochastic model, with typical values of the spontaneous emission noise strength, predict similar $\langle T_{\mathrm{LFF}}\rangle$ and PDF. Strong enough noise, however, induces escapes from the stable ECM, leading the laser output to display coexistence of LFFs and stable emission, similar to experimental observations [8].

We also show that the nonlinear gain saturation coefficient, ε, which is included phenomenologically in the LK model to represent a variety of saturation mechanisms such as carrier heating, carrier diffusion, and spatial hole burning, is a key parameter in determining the duration of the LFF transient: when increasing ε both the average transient time $\langle T_{\mathrm{LFF}}\rangle$ and the probability of noise-induced escapes from the stable ECM increase. Therefore, our results suggest that the LFFs observed experimentally can be, at least in part, sustained by the interplay of noise and the various nonlinear effects which are phenomenologically represented by the gain saturation coefficient.

Some characteristics of the LFF dynamics, in particular, the statistical properties of time intervals between power dropouts, can be explained by a rate-equation model proposed by Eguia, Mindlin, and Giudici (EMG model) [26], which supports the scenario that the laser behaves as an excitable system and that the LFFs are induced by noise. The dependence of the shape of the PDF of inter-dropout intervals on the pump current or the feedback strength was shown to be equivalent to variations produced by the two parameters of the EMG model [27]. Moreover, in [28] the periodic "spike" patterns generated experimentally under external periodic forcing were compared with the solutions of the EGM model, and it was shown that the topological organization of the experimentally observed periodic orbits was equivalent, in the parameter region explored, to the one displayed by the model solutions. The limits of the excitable LFF behavior, and thus the region of validity of the EMG model, were studied in [29], where it was shown that excitability deteriorates in the parameter region where there is a high probability of stable emission; in this region of "coexistence" the laser dynamics can be separated into stable and bursting states.

The EMG model is low dimensional, and thus it offers a good control to test the influence of the delay-induced high dimensionality of the LK model on the statistics of the LFF transient time. Thus in this article we also investigate the transient dynamics predicted by the EMG model with parameters in the excitable region. We show that by choosing appropriate initial conditions, a qualitatively good agreement is found with the features of the transient dynamics predicted by the LK model. However, in this case noise does affect the shape of the PDF of LFF transient times, which indicates the importance of the high dimensionality of the nonattracting chaotic set of the LK model in determining the transient time statistics. The article is organized as follows: Section II describes the LK model and discusses the initial conditions chosen for the simulations. Sections III–VI present the numerical results and discuss the statistical features of the LFF transient lifetime in terms of various parameters. Section VII presents the results of the control simulation of the low-dimensional EMG model. Finally, Section VIII contains a summary of results and the conclusions.

II. MODELING FRAMEWORK

We first describe the dynamics of a single-mode semiconductor laser with optical feedback by means of the well-known Lang-Kobayashi (LK) delay-differential rate equations for the slowly varying complex amplitude of the electric field, E, and the carrier density, N [16]:

$$\frac{dE}{dt} = k(1+i\alpha)[G(E,N)-1]E + \kappa_{\mathrm{fb}}E(t-\tau)e^{-i\omega_0\tau} + \sqrt{D}\xi(t), \tag{1}$$

$$\frac{dN}{dt} = \gamma_N[J - N - G(E,N)|E|^2], \tag{2}$$

where k is the field decay rate, α is the linewidth enhancement factor, and G is the optical gain, given by $G(E, N) = N/(1 + \varepsilon|E|^2)$, with ε being the gain saturation coefficient. γ_N is the carrier decay rate and J is the injection current parameter (normalized such that the solitary threshold current is $J_{th} = 1$ if $\varepsilon = 0$). The feedback term is characterized by its strength κ_{fb}, its associated delay time τ, and the feedback phase $\omega_0\tau$. $\xi(t)$ is an uncorrelated complex Gaussian white noise representing spontaneous emission fluctuations of strength D.

To integrate the model rate equations, Eqs. (1) and (2), we have to specify the initial conditions, which we choose to correspond to the steady state of the solitary laser plus a small random term:

$$E(t) = E_s e^{i\phi_0} + \eta\xi(t), \quad -\tau \leqslant t \leqslant 0, \tag{3}$$
$$N(0) = N_s + \rho\zeta, \tag{4}$$

where E_s and N_s are the stationary solutions of the system, which correspond, for a normalized injection current J below the solitary laser threshold, to the off state, $E_s = 0$ and $N_s = J$, whereas if $J > J_{\mathrm{th}}$ they correspond to the ECMs. Defining $E(t) = E_s e^{i(\omega_s-\omega_0)t}$ and $N(t) = N_s$, then the values of E_s, ω_s, and N_s are determined by

$$\omega_s\tau = \omega_0\tau - \kappa_{\mathrm{fb}}\tau\sqrt{1+\alpha^2}\sin(\omega_s\tau + \tan^{-1}\alpha), \tag{5}$$
$$N_s = \frac{1 - (\kappa_{\mathrm{fb}}/k)\cos(\omega_s\tau)}{(1+\varepsilon)} + \frac{J\varepsilon}{1+\varepsilon}, \tag{6}$$
$$|E_s|^2 = \frac{J - N_s}{N_s - (J - N_s)\varepsilon}. \tag{7}$$

Alternatively, one could always choose as initial condition the off state of the laser, regardless of the value of the injection current. As discussed later, we find that the results are robust with respect to the specific choice of the initial condition. Unless otherwise explicitly stated, we integrated the LK model with the parameter values given in Table I, using the stochastic Heun method with an integration time step of 0.8 ps. The simulations were verified using smaller integration steps and the Euler integration method, with which we obtained similar results.

TABLE I. Typical parameter values of the LK model described by Eqs. (1) and (2).

Description	Symbol	Value
Linewidth enhancement factor	α	3
Field decay rate	k	300 ns^{-1}
Feedback strength	κ_{fb}	30 ns^{-1}
External round-trip time	τ	6.667 ns
Feedback phase	$\omega_0\tau$	0 rad
Carrier population decay rate	γ_N	1 ns^{-1}
Normalized injection current	J	1.02
Gain saturation coefficient	ε	0
Spontaneous emission noise strength	D	10^{-4} ns^{-1}
Noise intensity (field initial condition)	η	10^{-3}
Noise intensity (carriers initial condition)	ρ	10^{-3}

III. TRANSIENT TIME DISTRIBUTION

As discussed in Sec. I, simulations of the LK model show that, close to the solitary laser threshold and with moderately strong optical feedback, the laser intensity displays fast picosecond pulses, which when subjected to a low-pass filter (as occurs in experiments, where photodetectors have a limiting bandwidth) transform into a collection of sudden dropouts, characteristic of the LFF dynamics, as shown in Fig. 1(a). All through this section a filter with a cutoff frequency of 120 MHz is applied to the intensity time trace, given by $|E(t)|^2$.

With initial conditions such that the laser is emitting on the stable state without feedback, at $t = 0$ the optical feedback is turned on. As a result the laser begins to experience intensity dropouts during a certain time interval $0 < t < T_{\mathrm{LFF}}$, as shown in Fig. 1(a). For $t \geqslant T_{\mathrm{LFF}}$ the laser output is stable, since the trajectory falls into the basin of attraction of one of the stable ECMs [fixed points given by Eqs. (5)–(7)], and remains trapped there provided the noise strength is not too large.

The lifetime of the transient LFF dynamics, T_{LFF}, is defined as the time interval during which the intensity fluctuations,

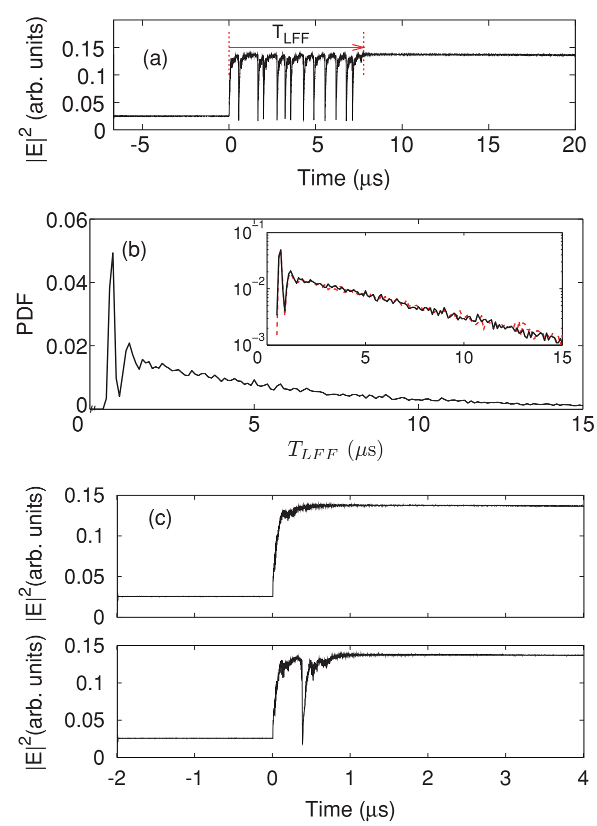

FIG. 1. (Color online) (a) Filtered intensity time trace obtained by integrating the LK model with parameters given by Table I. Typical LFF dropouts can be observed. The red arrow defines the transient time, T_{LFF}, which in this case is about 7.77 μs. (b) Probability distribution function (PDF) of the transient time T_{LFF} calculated from 30 000 realizations of the stochastic initial condition. Inset: PDF with vertical logarithmic scale to show the exponential tails with noise (solid line) and without noise (dashed line, red online). (c) Intensity time traces corresponding to the two maxima of the PDF shown in panel (b).

measured as the standard deviation calculated in a time window, ΔT, are above a certain threshold, chosen here to be 2% of the average intensity. To make sure that the system has reached an asymptotic behavior (in the vicinity of a fixed point), we use a time window of $\Delta T = 1800$ ns, much larger than the characteristic time scale of the fast intensity pulsations. The total integration time is of the order of 10–100 ms, which thus correspond to the longest transient times that we can compute.

The duration of the LFF transient depends on the specific realization of the random initial condition and can strongly deviate from its mean value. Figure 1(b) displays the (PDF) of the transient time T_{LFF}. The shape of this distribution can be understood as follows: the system has a zero probability of finding a stable ECM in a very short transient time T_{LFF}, due to the finite amount of time it takes to go from the initial condition (near the solitary laser's steady state) to the phase space region where the stable ECMs are located. The largest peak in the PDF corresponds to this single-rise travel time, which we refer to as T_1 (typically, $T_1 < 1$ μs depending on parameters). We show this trajectory in the top trace of Fig. 1(c). Note that there is a large probability that the system finds a stable ECM the first time it is in the region of the phase space where the stable ECMs are located. The secondary maximum of the PDF (T_2) corresponds to trajectories in which the system finds a stable ECM during its second visit to the area near it. In this case the transient dynamics contains one dropout, as shown in the bottom trace of Fig. 1(c).

In between T_1 and T_2 the system has a small probability of finding a stable ECM because it is in another region of the phase space (i.e., in the recovery process after the dropout). For larger values of T_{LFF} the PDF decays exponentially, as is expected in chaotic transients [1]. The inset of Fig. 1(b) plots the PDF in both the presence and absence of noise. The two distributions overlap, which suggests that the average transient time, $\langle T_{\mathrm{LFF}}\rangle$, is not affected by noise. We verify this fact in Fig. 2(a), which shows the average duration of the transition from LFFs to a stable output as a function of the noise strength, D, for different values of the injection current and feedback strength. In all cases the average transient time is not significantly affected by noise and is approximately equal to that of the noise-free case.

Later in this article it is shown that the T_{LFF} distribution strongly depends on the other laser parameters α and ε, besides J; therefore, it could be expected that for different values of these parameters the T_{LFF} distribution is not so insensitive to noise. To check this point we performed extensive simulations for other values of α and ε, and we present in Figs. 2(b) and 2(c) two examples of the results. Again, it can be observed that the average duration of the transient time does not significantly change with noise strength. Therefore, at least in the parameter region explored, we can conclude that the duration of the transient dynamics is not qualitatively affected by random fluctuations. It is important to remark that we have limited ourselves to explore the parameter region where the average transient time is not too long; for larger values of J, α, or τ the simulations require too long and unpractical computational times. Therefore, we cannot exclude that for larger values of J, α, or τ the noise has an effect on the transient time.

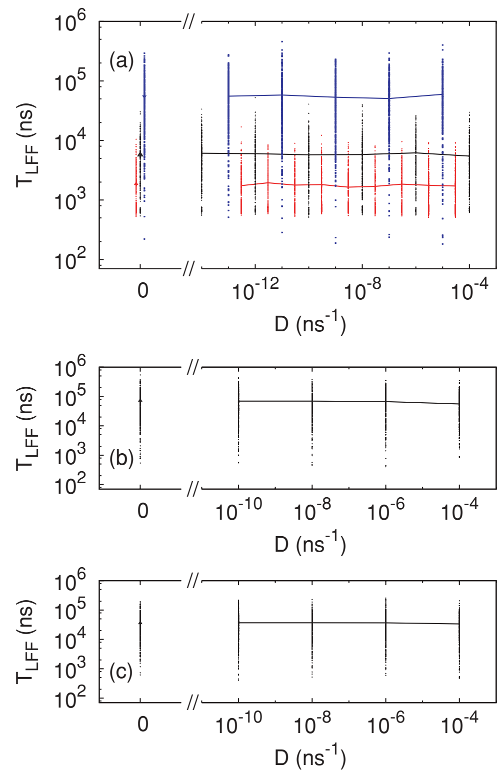

FIG. 2. (Color online) Transient time (dots) and average transient time $\langle T_{\mathrm{LFF}}\rangle$ (solid lines) for 300 realizations of the stochastic initial conditions as a function of the noise intensity for different values of the injection current and the feedback strength. (a) $\varepsilon = 0.0$, $\alpha = 3$, and, additionally, from top to bottom: $J = 1.02$, $\kappa_{\mathrm{fb}} = 15$ ns^{-1} (blue); $J = 1.02$, $\kappa_{\mathrm{fb}} = 30$ ns^{-1} (black) $J = 0.98$, $\kappa_{\mathrm{fb}} = 30$ ns^{-1} (red). (b) $\varepsilon = 0.06$, $\alpha = 3$, $J = 1.02$, $\kappa_{\mathrm{fb}} = 30$ ns^{-1}. (c) $\varepsilon = 0.1$, $\alpha = 2.6$, $J = 1.02$, $\kappa_{\mathrm{fb}} = 30$ ns^{-1}. Other parameters are as in Table I.

IV. EFFECT OF THE LASER PARAMETERS

To investigate how the LFF lifetime depends on the parameters of the system, we computed the average transient time, $\langle T_{\mathrm{LFF}}\rangle$, for varying values of different parameters, classified in terms of laser parameters (this section) and optical feedback parameters (next section). An interesting effect is provided by the gain saturation coefficient, ε. When increasing ε in a realistic range the average transient time $\langle T_{\mathrm{LFF}}\rangle$ increases three orders of magnitude, as shown in Fig. 3(a). In fact, nonlinear gain saturation acts as a coupling between the field and the phase in a way similar to the linewidth enhancement factor, α, whose effect is displayed in Fig. 3(b).

Recently, Torcini *et al.* [25] analyzed the relationship between the stability of the ECMs and the length of the LFF transient and derived an analytical expression for estimating the transient time in relation to the eigenvalues of the stable ECMs. In the specific range of parameters examined in [25], $J < J_{\mathrm{th}}$ and $\alpha < 4$, a periodic variation of $\langle T_{\mathrm{LFF}}\rangle$ with α was found [see the inset in Fig. 3(b)], which was well understood in terms of the analytical expression derived. However, the agreement worsens for bias currents above the solitary threshold, which is the parameter range examined here.

The influence of the injection current parameter, J, is displayed in Fig. 3(c), which shows that the transient time

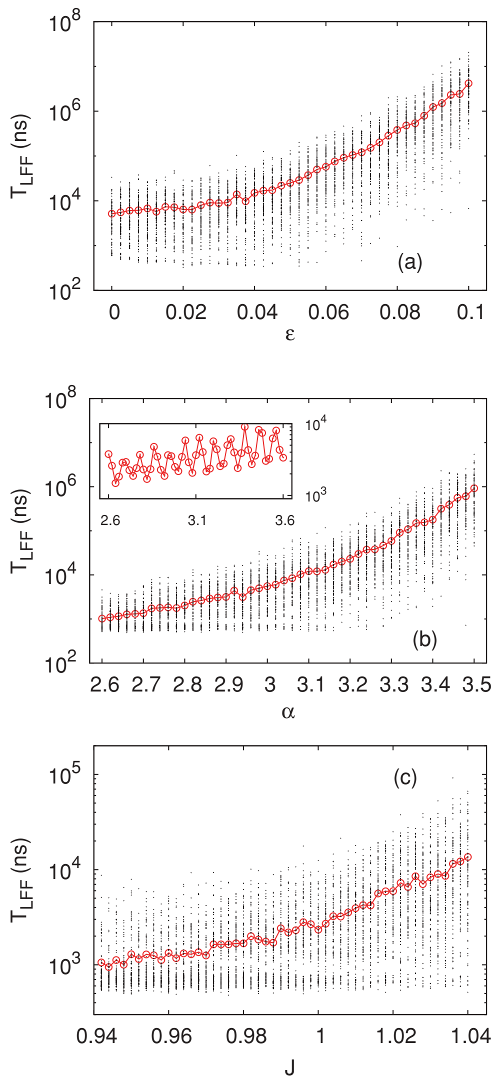

FIG. 3. (Color online) Transient time (dots) and average transient time $\langle T_{\rm LFF}\rangle$ (red circles) for 100 random realizations of the initial conditions, as a function of (a) the gain saturation coefficient with $\alpha = 3$ and $J = 1.02$, (b) the linewidth enhancement factor with $J = 1.02$ and $\varepsilon = 0$, and (c) the injection current with $\alpha = 3$ and $\varepsilon = 0$. In the inset in panel (b), we show results for the same parameters as in [25]. Other parameters are $\kappa_{\rm fb} = 30$ ns^{-1}, $\tau = 6.667$ ns, and $D = 10^{-4}$ ns^{-1}.

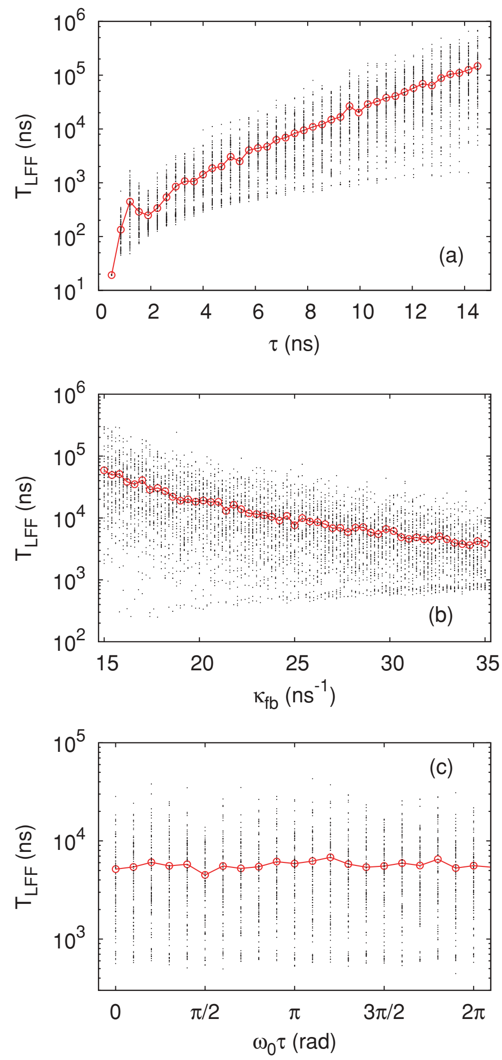

FIG. 4. (Color online) Transient time (dots) and average transient time $\langle T_{\rm LFF}\rangle$ (circles, red online) for 100 stochastic realizations of the initial conditions, as a function of (a) delay time with $\kappa_{\rm fb} = 30$ ns^{-1} and $\omega_0\tau = 0$, (b) the feedback strength with $\tau = 6.667$ ns and $\omega_0\tau = 0$, and (c) the feedback phase with $\tau = 6.667$ ns and $\kappa_{\rm fb} = 30$ ns^{-1}. Other parameters are $\varepsilon = 0$, $\alpha = 3$, $J = 1.02$, and $D = 10^{-4}$ ns^{-1}.

$\langle T_{\rm LFF}\rangle$ also increases with J. Our results are consistent with those in [25], where it was shown that the transient time increases with both α and J. These figures also show the existence of a minimum transient time, as discussed previously in relation to Fig. 1(b).

V. EFFECT OF THE OPTICAL FEEDBACK PARAMETERS

The influence of the delay time τ is depicted in Fig. 4(a). For small delays the dynamics is not chaotic and all realizations of the stochastic initial conditions lead to almost the same transient time. As we increase τ, the average transient time $\langle T_{\rm LFF}\rangle$ increases nearly exponentially up to the maximum delay studied. This is due to a nearly exponential increase of the average number of dropouts during the transient, while the mean time interval between dropouts increases *monotonically* with the delay [7,30].

For increasing feedback strength, $\kappa_{\rm fb}$, the duration $\langle T_{\rm LFF}\rangle$ of the transient decreases, as depicted in Fig. 4(b). Although the parameter region is different, it is interesting to compare this result with those of [24], where the authors show that for low feedback levels, that is, for a small number of ECMs, sustained and transient dynamics alternate for increasing $\kappa_{\rm fb}$.

Another delay parameter is the feedback phase, $\omega_0\tau$. By increasing $\omega_0\tau$, pairs of modes and anti-modes are created far from the chaotic attractor, and they are destructed in the region of phase space where the stable ECMs are. Varying $\omega_0\tau$ also changes the stability of the ECMs with a periodicity of 2π. Then it could be expected that at least one of the ECMs collides with the chaotic attractor and it may be reflected in $\langle T_{\rm LFF}\rangle$ with the same periodicity. Figure 4(c) shows that varying $\omega_0\tau$ does

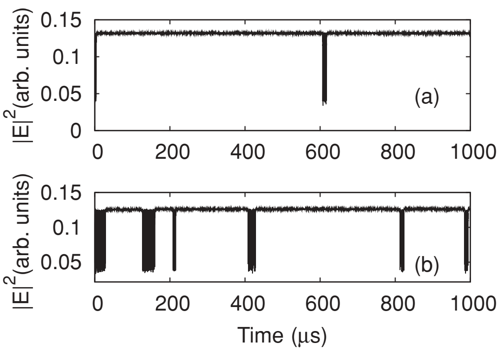

FIG. 5. (a) Intensity time trace for relatively large noise strength ($\varepsilon = 0$, $D = 10^{-2}$ ns^{-1}). (b) Same as in panel (a) with $\varepsilon = 0.05$.

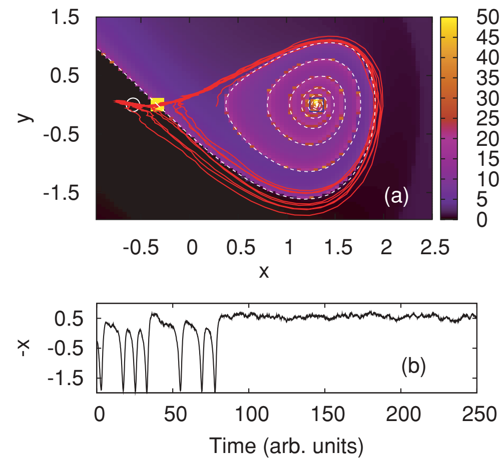

FIG. 6. (Color online) (a) Phase space portrait of the EMG model. For the parameters chosen the system is in an excitable regime, exhibiting a stable node (green circle), a saddle point (yellow full square), and a repeller (blue open square). The background color represents the transient time of a trajectory starting at that point in phase space in the absence of noise. The white dashed line represents the stable manifold of the saddle point, and the red solid line shows a typical trajectory with noise. (b) Time trace of the EMG model corresponding to the red trajectory shown in plot (a), for the variable $-x(t)$. The minus sign is chosen to compare with Fig. 1(a). Noise intensity is $d = 2 \times 10^{-3}$, and the deterministic parameters of the model are $\epsilon_1 = 0.25$ and $\epsilon_2 = 0.4$.

not change the average transient time in a significant way. This result indicates that the stabilities of the LFF dynamics and of the ECMs are not directly related, at least for the parameter values examined here, for which there are global trajectories in phase space.

VI. COMBINED EFFECT OF LARGE NOISE AND GAIN SATURATION

Even after the laser has settled around the stable ECM once the chaotic transient has finished, strong enough noise can lead the trajectory to eventually escape and display another set of LFF dropouts, as shown in Fig. 5(a). The ensuing transient LFF regime is similar to the one studied previously, in which the laser was off at $t = 0$, when the feedback was turned on. These two situations only differ in the choice of initial conditions, which as discussed previously lead to the same distribution of LFF durations. Noise-induced escape of the basin of attraction of the stable ECMs was studied in [25] for large enough noise and interpreted in terms of the Kramers rate. This provides the system with two time scales that can be tuned separately and could lead to resonant effects such as stochastic or coherent resonance. One of these time scales (the excursion duration) is deterministic, as shown in Fig. 2, and the other one (the escape time) is stochastic, as shown in [25]. Finally, we note that, if nonlinear gain saturation is included in the simulations, the probability of noise-induced escape away from the stable ECMs substantially increases, as shown in Fig. 5(b).

VII. TRANSIENTS IN A LOW-DIMENSIONAL PHENOMENOLOGICAL MODEL

Eguia, Mindlin, and Giudici proposed a phenomenological model (EMG model) that describes the dynamics of the *time-averaged* laser intensity, that is, not the fast picosecond pulses but the slower dropouts [26]. The model is defined by the following set of ordinary differential equations:

$$\frac{dx}{dt} = y + \sqrt{d}\xi(t), \tag{8}$$

$$\frac{dy}{dt} = x - y - x^3 + xy + \epsilon_1 + \epsilon_2 x^2, \tag{9}$$

where ϵ_1 and ϵ_2 are two control parameters, d is the noise strength, and $\xi(t)$ is a Gaussian white noise.

We chose a parameter for which the model operates in an excitable regime, with three fixed points (x_s, y_s) with $y_s = 0$ and x_s being a solution of the third order equation $x - x^3 + \epsilon_1 + \epsilon_2 x^2 = 0$. The three fixed points are a stable focus, a saddle point, and an unstable focus (repeller), shown as symbols in Fig. 6(a). In a previous work, Yacomotti *et al.* [27] associated the parameter ϵ_1 with the bias current and ϵ_2 with the feedback strength. Exploiting this similarity we chose the initial conditions as similar as possible to the ones described in the previous sections. Specifically, we chose random initial conditions for $\epsilon_2 = 0$ inside the region limited by the stable manifold of the saddle point and the repeller:

$$x(t = 0) = x_0 + r\xi, \tag{10}$$

$$y(t = 0) = y_0 + r\zeta, \tag{11}$$

where $x_0 = 0.4$, $y_0 = 0$, and $r = 0.25$. ξ and ζ are uncorrelated Gaussian random numbers. We integrated the EMG model using the stochastic Heun method with an integration time step of 8×10^{-3} arbitrary units. Some characteristics of the LFF dynamics can be satisfactorily reproduced by the EMG model. In particular, the transient regime can be reproduced approximately, as shown in Fig. 6(b). The distribution of transient times obtained in this case is plotted in Fig. 7 and shows a qualitative agreement with the results found in the aforementioned LK model. In this case, however, noise does play an important role, changing qualitatively the shape of the distribution for large transient times, as shown in the inset of Fig. 7. In order to understand how this distribution function arises and why noise plays a more important role in this case, we have examined the dependence of the transient time on the initial conditions for the deterministic

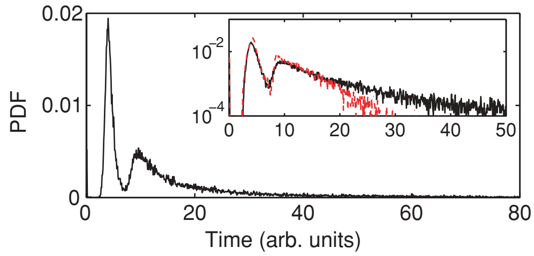

FIG. 7. (Color online) Probability distribution function for the transient time for the EMG model. Parameters are as in Fig. 6. Inset: the same plot with vertical logarithmic scale with noise (solid line) and without noise (red dashed line).

model. This dependence is shown in color coding in Fig. 6(a). The results presented in this figure reveal that the initial conditions leading to a given transient time have a well-defined structure in phase space, with the transient time being larger the closer the initial conditions are to the stable manifold of the saddle (white dashed line in the figure). In that case, a substantial slowdown is experienced by the trajectory as it passes nearby the saddle, leading to the exponential time in the transient time distribution. Noise seems to increase the probability that trajectories encounter this area of phase space, thus increasing the fraction of large transient times. Thus, in this region, the transient time depends strongly on the noise fluctuations, and unlike in the LK model [Fig. 1(b)], long transient times are induced by noise.

VIII. CONCLUSIONS

We have studied numerically the transient LFF dynamics of a semiconductor laser with optical feedback using the well-known LK model. We defined the transient time as the time taken by the intensity fluctuations to decrease below a chosen threshold, which occurs when the system leaves the chaotic LFF attractor and falls into one of the stable fixed points (the so-called ECMs). The PDF of the transient time has an exponential tail that is characteristic of chaotic transients, and there is a minimum transient time due to the finite amount of time needed to go from the fixed point of the solitary laser to one of the stable ECMs of the laser with feedback.

We found that in the LK model noise does not *significantly* affect the average transient time or its distribution for realistic parameter values. This demonstrates that the transient LFF is mainly a deterministic phenomenon, its duration being determined by the various model parameters that affect the time needed to go from the fixed point of the solitary laser to a stable ECM. We have also shown that sufficiently large values of the noise strength can induce escapes from the stable ECM, leading to regimes of power dropouts alternating with intervals of stable steady-state emission. This behavior provides evidence that transient LFFs are excitable due to the effect of noise.

We presented an in-depth analysis of the statistical properties of this transient dynamics and investigated the influence of different parameters. Our results show that the nonlinear gain saturation coefficient, which represents various gain saturation effects, plays a key role in determining the duration of the LFF lifetime: a small variation of the saturation coefficient results in a drastic increase of the duration of the LFF transient. Nonlinear gain saturation also increases the probability of noise-induced escapes, and therefore, our results suggest that the LFFs observed in experiments can be, at least in part, sustained by various nonlinear light-matter interactions in the laser active medium.

Finally, we have compared the behavior of the delay-differential LK model with that of a phenomenological ordinary differential equation (ODE) model [26] operating in the excitable regime and with appropriate initial conditions. This comparison shows that noise plays an important role in the transient dynamics when the dimensionality of the system is low, but not when it is large (due to the explicit delay in the LK model). It would be interesting to investigate whether this type of behavior also occurs in other high-dimensional chaotic systems.

ACKNOWLEDGMENTS

Stimulating discussions with J. Tiana-Alsina and M. C. Torrent are gratefully acknowledged. This research was supported in part by the US Air Force Office of Scientific Research under Grant FA9550-07-1-0238, the Spanish Ministerio de Educacion y Ciencia through Project FIS2009-13360, and the Agencia de Gestio d'Ajuts Universitaris i de Recerca (AGAUR), Generalitat de Catalunya, through Project 2009 SGR 1168.

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