Quantifying noise-induced order and noise-induced complexity via information theory measures

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Instabilities and Non-equilibrium Structures XII
On the occasion of the 60th birthday of Pierre Coullet
Viña del Mar, December 14th -18th 2009
Quantifying noise-induced order and noise-induced complexity via information theory measures

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- It can be easy to recognize; but difficult to define quantitatively.

- Complexity is characterized by signatures of spatial and/or temporal correlational structures.

- We would like to find a quantity “$C$” that measures complexity, as the entropy, “$H$”, measures randomness, and, for low-dimensional systems, the Lyapunov exponents, $\lambda_i$, measure chaos.
The simple and the complex

Order

\[
\begin{align*}
H &= 0 \\
C &= 0
\end{align*}
\]

Chaos

\[
\begin{align*}
H &\neq 0 \\
C &\neq 0
\end{align*}
\]

Disorder

\[
\begin{align*}
H &= 1 \\
C &= 0
\end{align*}
\]
Complexity?

So, one can begin by excluding processes that are not complex:
- periodic motion
- a purely random process

In between “perfect order” (E.g., regular crystal) and “complete disorder” (E.g., ideal gas), complexity can be characterized by a certain degree of organization, structure, memory, regularity, symmetry, and patterns.

A useful complexity measure needs to do more than satisfy the boundary conditions of vanishing in the high- and low-entropy limits.

Maximum complexity occurs in the region between the system’s perfectly ordered state and the perfectly disordered one.

Feldman, McTague and Crutchfield, Chaos 2008

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Can complexity be induced by noise?

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Outline of the talk

- **Introduction**
  - Complexity quantifiers
  - Noise-induced order (stochastic resonance and coherence resonance)

- Noise-induced complexity

- **Summary & Conclusions**
Outline of the talk

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- Summary & Conclusions
Assuming that we know the probability distribution $P=[p_i, \ i=1,N]$ that fully characterizes a given system, we can define the an information measure

- Shannon entropy
  $$I[P] = S_s[P] = -\sum_i p_i \ln p_i$$

- Tsallis entropy
  $$I[P] = S_T^q[P] = \frac{1}{q-1} \left[ 1 - \sum_i p_i^q \right]$$

- Renyi entropy
  $$I[P] = S_R^q[P] = \frac{1}{1-q} \ln \left[ \sum_i p_i^q \right]$$
Normalized information \( H \)

\[
H[P] = \frac{I[P]}{I_{\text{max}}} \quad 0 \leq H[P] \leq 1
\]

where

\[
I_{\text{max}} = I[P_e]
\]

\( P_e \) being the \textit{equilibrium} probability distribution (that maximizes the information measure).

**Example:** if \( I[P] = \text{Shannon entropy} = S_S[P] \)

then \( P_e = [p_i=1/N \text{ for } i=1,N] \)

and \( I_{\text{max}} = \ln(N) \)
Disequilibrium $Q$

Measures the "distance" from $P$ to the equilibrium distribution, $P_e$

$$Q[P] = Q_0 D[P, P_e]$$

where $Q_0$ is a normalization constant such that $0 \leq Q[P] \leq 1$
Distance between $P$ and $P_e$

- **Euclidean**

$$D_E[P, P_e] = \left\| P - P_e \right\|_E = \sum_i (p_i - 1/N)^2$$

- **Wootters**

$$D_W[P, P_e] = \cos^{-1}\left[ \sum_i p_i^{1/2} \left(1/N\right)^{1/2} \right]$$

- **Kullback relative entropy**


- **Jensen divergence**

$$D_J[P, P_e] = \frac{K[P | P_e] + K[P_e | P]}{2}$$

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A family of complexity measures can be defined as:


where

- \( A = S, T, R \) (Shannon, Tsallis, Renyi)
- \( B = E, W, K, J \) (Euclidean, Wootters, Kullback, Jensen)

\[ C_{LMC}[P] = H_S[P] \cdot Q_E[P] \]

\[ C_{MPR}[P] = H_S[P] \cdot Q_J[P] \]


The complexity is a non-trivial function of the entropy: for a given value of $H$, there is a range of possible values of $C$.

$C=0$ if $H=0$ or $H=1$.

Also, complexity-entropy diagrams have shown to give a reliable way to distinguish between “normal” brains and those experiencing cortical thinning, a condition associated with Alzheimer’s disease.

How to define a probability distribution that fully characterizes the correlational structures present in a given system?
Given a time series: \( X = \{ x_t, \ t = 1 \ldots M \} \)

We can associate a probability distribution based on:

- Histogram of amplitudes
- Binary representation (symbolic dynamics)
- Frequency (Fourier transform)
- Frequency bands (Wavelet transform)
- Ordinal Patterns (attractor representation)
Example: The Logistic Map

Binary representation:
0 if $x \leq 1/2$, 1 if $x > 1/2$

Strings of length 12 were considered to represent states of the system.

The probabilities were assigned according to the frequency of occurrence in $2^{22}$ interactions.
The complexity of the Logistic map

calculated with the Lempel & Zip complexity measure

Kaspar and Schuster, PRA 1987
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Probability distribution from ordinal patterns

Attractor representation based on the comparison of consecutive values

For patterns of order D, # of ordinal patterns is $D!$

Good statistics when # of data points: $M \gg D!$

Brandt & Pompe, PRL 2002

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- Summary, Conclusions & Future Work

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Stochastic Resonance

\[ \dot{x} = x - x^3 + A \cos(\omega t) + \sqrt{2D} \xi \]

\[ \langle \xi(t)\xi(0) \rangle = \delta(t) \]

bistable system + subthreshold signal + noise

Varying D keeping \( \omega \) fixed

Varying \( \omega \) keeping D fixed


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Quantifying SR

- Signal-to-noise ratio
- Histogram of residence times

Varying $D$ ($\omega$ fixed)

Varying $\omega$ ($D$ fixed)

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Coherence Resonance

\[
\begin{align*}
\frac{dx}{dt} &= x - \frac{x^3}{3} - y, \\
\frac{dy}{dt} &= x + a + D\xi(t).
\end{align*}
\]

- FitzHugh-Nagumo excitable system + noise

- Normalized variance of inter-spike intervals
  \[
  R_p = \frac{\sqrt{\text{Var}(t_p)}}{\langle t_p \rangle}
  \]

- Correlation time
  \[
  \tau_c = \int_0^\infty C^2(t) \, dt
  \]

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- Introduction

- Noise-induced complexity
  - SR & CR (numerical simulations)
  - Semiconductor laser with optical feedback (experiments)

- Summary & Conclusions
Stochastic resonance

Bandt and Pompe method applied to consecutive residence times intervals.

Varying the noise strength ($\omega$ fixed) $D=6, M=60000$

empty symbols: data rearranged randomly

Rosso & Masoller, PRE 79, 040106(R) (2009)
Rosso & Masoller, EPJB 69 (2009)
Special issue on Stochastic Resonance

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Stochastic resonance

Varying $\omega$ (D fixed)

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empty symbols: data rearranged randomly
Coherence resonance

Bandt and Pompe method applied to consecutive inter-spike intervals

\[ \varepsilon \frac{dx}{dt} = x - \frac{x^3}{3} - y, \]
\[ \frac{dy}{dt} = x + a + D \xi(t). \]

solid symbols: \( \varepsilon = 0.01, \ a = 1.05; \)
empty symbols: \( \varepsilon = 0.1, \ a = 1.005 \)

Rosso & Masoller, PRE 79, 040106(R) (2009)
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- Summary & Conclusions
Low Frequency Fluctuations (LFFs) in semiconductor lasers with optical feedback

Photo-detector + oscilloscope signal

Streak camera signal

The intensity dropouts are the envelope of fast pulses

LFFs occur when the laser is biased close to threshold

I. Fischer et al., PRL 76, 220 (1996)
Torcini et al, PRA 74, 063801 (2006)
CR and SR and LFFs

FIG. 1. Temporal behavior of the laser intensity for increasing input noise amplitude. From top to bottom: noise = $-60.8 \text{ dBm/ MHz}$ (a), $-52.5 \text{ dBm/ MHz}$ (b), and $-44.3 \text{ dBm/ MHz}$ (c). The horizontal scale is 100 ns/div. The vertical scale is the same for the three plots.

Giacomelli, Giudici, Balle and Tredicce, PRL 2000

Marino et al, PRL 2002

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**Deterministic Coherence Resonance**

Injection current

Coherence collapse (fully developed chaos)

Heil et al, PRA 58, R2672 (1998)

\[ R_p = \frac{\sqrt{\text{Var}(t_p)}}{\langle t_p \rangle} \]

FIG. 6. Experimental normalized variance of the time between LFF drops as a function of the injection current, for a feedback delay of 6 ns.

Martinez-Avila, de S. Cavalcante and Rios Leite PRL 2004

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Experiments (DONLL in Terrassa)

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Time series analysis (O. A. Rosso in Australia)

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![Graphs showing time series analysis results](image-url)
Outline of the talk

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- Noise-induced complexity
- Summary & Conclusions
We have shown that the normalized Shannon entropy and the statistical complexity can be employed to detect and quantify resonant-like behavior in the form of enhanced temporal order induced by the variation of a system parameter or by the variation of the noise strength.

The success of the method is based on an appropriate reconstruction of the attractor and on an appropriate partition of the phase space that results in ordinal patterns having a probability distribution that fully characterizes the correlations in the system.

We applied the method to the analysis of numerical and experimental data:
- Residence time intervals in the bistable system (SR).
- Inter-spike intervals in the FHN model (CR).
- Inter-dropout intervals of the LFF regime (exp).