Nonlinear systems, chaos and control in Engineering

Introduction to Dynamical Systems, fixed points, linear stability analysis, and numerical integration

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- **Introduction to dynamical systems**
- Introduction to flows on the line
- Fixed points and linear stability
- Solving equations with computer
- Systems that evolve in time.
- Examples:
  - Pendulum clock
  - Neuron
- Dynamical systems can be:
  - linear or nonlinear (harmonic oscillator – pendulum);
  - deterministic or stochastic;
  - low or high dimensional;
  - continuous time or discrete time.

In this course: nonlinear systems (Nonlinear Dynamics)
• After a transient the systems settles down to equilibrium (rest state or “fixed point”).

• Keeps spiking in cycles (“limit cycle”).

• More complicated: chaotic or complex evolution (“chaotic attractor”).
Introduction to dynamical systems
Mid-1600s: Ordinary differential equations (ODEs)

**Isaac Newton**: studied planetary orbits and solved analytically the “two-body” problem (earth around the sun).

Since then: a lot of effort for solving the “three-body” problem (earth-sun-moon) – Impossible.
Henri Poincare (French mathematician). Instead of asking “which are the exact positions of planets (trajectories)?” he asked: “is the solar system **stable** for ever, or will planets eventually run away?”

- He developed a **geometrical** approach to solve the problem.
- Introduced the concept of “phase space”.
- He also had an **intuition** of the possibility of **chaos**.
Deterministic system: the initial conditions fully determine the future state. **There is no randomness but the system can be unpredictable.**
Computes allowed to experiment with equations.

Huge advance of the field of “Dynamical Systems”.


Chaotic motion.
Order within chaos and self-organization

- **Ilya Prigogine** (Belgium, born in Moscow, Nobel Prize in Chemistry 1977)
- Thermodynamic systems far from equilibrium.
- Discovered that, in chemical systems, the interplay of (external) **input of energy** and **dissipation** can lead to “self-organized” patterns.

In the 1960’s he did experiments trying to understand the effects of perturbations in biological clocks (circadian rhythms).

What is the effect of an external perturbation on subsequent oscillations?
Robert May (Australian, 1936): population biology


\[ x_{t+1} = f(x_t) \]

Example: \( f(x) = r x(1 - x) \)

Difference equations (“iterated maps”), even though simple and deterministic, can exhibit different types of dynamical behaviors, from stable points, to a bifurcating hierarchy of stable cycles, to apparently random fluctuations.
The logistic map

\[ x(i + 1) = r \cdot x(i) [1 - x(i)] \]

r = 2.8, Initial condition: \( x(1) = 0.2 \)

Transient relaxation → long-term stability

"period-doubling" bifurcations to chaos

Transient dynamics → stationary oscillations (regular or irregular)
In 1975, Mitchell Feigenbaum (American mathematical physicist), using a small HP-65 calculator, discovered the scaling law of the bifurcation points:

$$\lim_{n \to \infty} \frac{r_{n-1} - r_{n-2}}{r_n - r_{n-1}} = 4.6692...$$

Then, he showed that the same behavior, with the same mathematical constant, occurs within a wide class of functions, prior to the onset of chaos (universality).

Very different systems (in chemistry, biology, physics, etc.) go to chaos in the same way, quantitatively.
Benoit Mandelbrot (Polish-born, French and American mathematician 1924-2010): “self-similarity” and fractal objects:

each part of the object is like the whole object but smaller.

Because of his access to IBM's computers, Mandelbrot was one of the first to use computer graphics to create and display fractal geometric images.
Are characterized by a “fractal” dimension that measures roughness.

- Broccoli: $D = 2.66$
- Human lung: $D = 2.97$
- Coastline of Ireland: $D = 1.22$

In the 80’s: can we observe chaos experimentally?

- **Optical chaos**: first observed in laser systems.
In the 90’s: can we control chaotic dynamics?

- Ott, Grebogi and Yorke (1990)
  Unstable periodic orbits can be used for control: wisely chosen periodic kicks can maintain the system near the desired orbit.

- Pyragas (1992)
  Control by using a continuous self-controlling feedback signal, whose intensity is practically zero when the system evolves close to the desired periodic orbit but increases when it drifts away.
Experimental demonstration of control of optical chaos
Unidirectionaly coupled Lorenz systems: the ‘x’ variable of the response system is replaced by the ‘x’ variable of the drive system.

$t \rightarrow \infty \quad |y_2 - y_1| \rightarrow 0, \quad |z_2 - z_1| \rightarrow 0$
Different types of synchronization

- **Complete (CS):** $x_1(t) = x_2(t)$ (identical systems)
- **Phase (PS):** the phases of the oscillations synchronize, but the amplitudes are not.
- **Lag (LS):** $x_1(t+\tau) = x_2(t)$
- **Generalized (GS):** $x_2(t) = f(x_1(t))$ (f depends on the strength of the coupling)

A lot of work is being devoted to detect synchronization in real-world data.
Experimental observation of synchronization in coupled lasers

Synchronization of a large number of coupled oscillators

Figure 1 | Fireflies, fireflies burning bright. In the forests of the night, certain species of firefly flash in perfect synchrony — here *Pteroptyx malaccae* in a mangrove apple tree in Malaysia. Kaka *et al.*² and Mancoff *et al.*³ show that the same principle can be applied to oscillators at the nanoscale.
Kuramoto model  
(Japanese physicist, 1975)

Model of all-to-all coupled phase oscillators.

\[ \frac{d\theta_i}{dt} = \omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i) + \xi_i, \quad i = 1...N \]

K = coupling strength, \( \xi_i \) = stochastic term (noise)

Describes the emergence of collective behavior

How to quantify?

With the order parameter:

\[ re^{i\psi} = \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j} \]

\( r = 0 \) incoherent state (oscillators scattered in the unit circle)
\( r = 1 \) all oscillators are in phase (\( \theta_i = \theta_j \ \forall \ i,j \))
Synchronization transition as the coupling strength increases

Strogatz and others, late 90’s

Strogatz, Nature 2001

Video: https://www.ted.com/talks/steven_strogatz_on_sync
Interest moves from chaotic systems to complex systems (small vs. very large number of variables).

Networks (or graphs) of interconnected systems

**Complexity science**: dynamics of emergent properties
- Epidemics
- Rumor spreading
- Transport networks
- Financial crises
- Brain diseases
- Etc.
The challenge: to understand how the network **structure** and the **dynamics** (of individual units) determine the collective behavior.

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Strogatz
Nature 2001,
Dynamical systems allow to
- understand low-dimensional systems,
- uncover “order within chaos”,
- uncover universal features
- control chaotic behavior.

Complexity science: understanding emerging phenomena in large sets of interacting units.

Dynamical systems and complexity science are interdisciplinary research fields with many applications.
- Introduction to dynamical systems
- **Introduction to flows on the line**
- Solving equations with computer
- Fixed points and linear stability
Types of dynamical systems

- **Continuous time**: differential equations
  - Ordinary differential equations (ODEs).
    Example: damped oscillator
    \[
    m \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + kx = 0
    \]
  - Partial differential equations (PDEs).
    Example: heat equation
    \[
    \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}
    \]

- **Discrete time**: difference equations or “iterated maps”. Example: the logistic map
  \[
  x(i+1) = r \cdot x(i)[1-x(i)]
  \]
ODEs can be written as **first-order** differential equations

\[
m \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + kx = 0
\]

\[
\Rightarrow
\begin{align*}
\dot{x}_1 &= f_1(x_1, \ldots, x_n) \\
& \vdots \\
\dot{x}_n &= f_n(x_1, \ldots, x_n)
\end{align*}
\]

- First example: harmonic oscillator

\[
x_1 = x \quad \text{and} \quad x_2 = \dot{x}
\]

\[
\dot{x}_2 = \ddot{x} = -\frac{b}{m} \dot{x} - \frac{k}{m} x = -\frac{b}{m} x_2 - \frac{k}{m} x_1
\]

- Second example: pendulum

\[
\ddot{x} + \frac{g}{L} \sin x = 0
\]

\[
\Rightarrow
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{g}{L} \sin x_1
\end{align*}
\]
Given the initial conditions, $x_1(0)$ and $x_2(0)$, we predict the evolution of the system by solving the equations: $x_1(t)$ and $x_2(t)$.

$x_1(t)$ and $x_2(t)$ are solutions of the equations.

The evolution of the system can be represented as a trajectory in the phase space.

⇒ two-dimensional (2D) dynamical system.

Key argument (Poincare): find out how the trajectories look like, without solving the equations explicitly.

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{b}{m}x_2 - \frac{k}{m}x_1
\end{align*}
\]
Classification of dynamical systems described by ODEs (I/II)

\[ \dot{x} = f(x) + \xi(t) \]

- **f(x) linear:** in the function \( f \), \( x \) appears to first order only (no \( x^2, x_1x_2, \sin(x) \) etc.). Then, the behavior can be understood from the sum of its parts.
- **f(x) nonlinear:** superposition principle fails!
- **Example of linear system:** harmonic oscillator
  \[
  m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = 0 \quad \Rightarrow \quad \begin{align*}
  \dot{x}_1 &= x_2 \\
  \dot{x}_2 &= -\frac{b}{m} x_2 - \frac{k}{m} x_1
  \end{align*}
  \]
  In the right-hand-side \( x_1 \) and \( x_2 \) appear to first power (no products etc.)
- **Example of nonlinear system:** pendulum
  \[
  \ddot{x} + \frac{g}{L} \sin x = 0 \quad \Rightarrow \quad \begin{align*}
  \dot{x}_1 &= x_2 \\
  \dot{x}_2 &= -\frac{g}{L} \sin x_1
  \end{align*}
  \]
Classification of dynamical systems described by ODEs (II/II)

\[
\dot{x} = f(x) + \xi(t)
\]

- \(\xi = 0\): deterministic.
- \(\xi \neq 0\): stochastic (real life) –simplest case: additive noise.
- \(x\): vector with few variables \((n<4)\): low dimensional.
- \(x\): vector with many variables: high dimensional.
- \(f\) does not depend on time: autonomous system.
- \(f\) depends on time: non-autonomous system.
Example of non-autonomous system: a forced oscillator

\[ m \ddot{x} + b \dot{x} + kx = F \cos t \]

- Can also be written as first-order ODE

\[
\begin{align*}
  x_1 &= x \quad \text{and} \quad x_2 = \dot{x} \\
  x_3 &= t \quad \dot{x}_3 = 1
\end{align*}
\]

\[
\begin{align*}
  \dot{x}_1 &= x_2 \\
  \dot{x}_2 &= \frac{1}{m}(-kx_1 - bx_2 + F \cos x_3) \\
  \dot{x}_3 &= 1
\end{align*}
\]

- Three-dimensional system: to predict the evolution we need to know the present state \((t, x, dx/dt)\).
So…what is a “flow on the line”?

- A one-dimensional autonomous dynamical system described by a first-order ordinary differential equation

\[ \dot{x} = f(x) \]

- \( x \in \mathbb{R} \)

- \( f \) does not depend on time
## Number of variables

<table>
<thead>
<tr>
<th>Linear</th>
<th>Nonlinear</th>
<th>N=1</th>
<th>N=2</th>
<th>N=3</th>
<th>N&gt;&gt;1</th>
<th>N=∞ (PDEs DDEs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>RC circuit</td>
<td>Logistic population grow</td>
<td>Harmonic oscillator</td>
<td>Pendulum</td>
<td>Forced oscillator</td>
<td>Kuramoto phase oscillators</td>
<td>Heat equation, Maxwell equations, Schrodinger equation</td>
</tr>
</tbody>
</table>

### Equations

\[
\frac{dx(t)}{dt} = Ax(t) + Bu(t) \\
x(t) = (x_1(t), \ldots, x_N(t))
\]

**PDEs** = partial differential eqs.  
**DDEs** = delay differential eqs.

“flow on the line”
Outline

- Introduction to dynamical systems
- Introduction to flows on the line
- **Solving equations with computer**
- Fixed points and linear stability
- Euler method

\[ x(t_0 + \Delta t) = x_1 = x_0 + f(x_0)\Delta t \]
\[ x_{n+1} = x_n + f(x_n)\Delta t \]
\[ t_n = t_0 + n\Delta t \]

- Euler second order

\[ \tilde{x}_{n+1} = x_n + f(x_n)\Delta t \]
\[ x_{n+1} = x_n + \frac{1}{2}[f(x_n) + f(\tilde{x}_{n+1})]\Delta t \]
Fourth order (Runge-Kutta 1905)

\[
\begin{align*}
k_1 &= f(x_n) \Delta t \\
k_2 &= f(x_n + \frac{1}{2} k_1) \Delta t \\
k_3 &= f(x_n + \frac{1}{2} k_2) \Delta t \\
k_4 &= f(x_n + k_3) \Delta t.
\end{align*}
\]

\[
x_{n+1} = x_n + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4)
\]

Problem if \( \Delta t \) is too small: round-off errors (computers have finite accuracy).
Table 12.1. *MATLAB’s ODE solvers.*

<table>
<thead>
<tr>
<th>Solver</th>
<th>Problem type</th>
<th>Type of algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>ode45</td>
<td>Nonstiff</td>
<td>Explicit Runge–Kutta pair, orders 4 and 5</td>
</tr>
<tr>
<td>ode23</td>
<td>Nonstiff</td>
<td>Explicit Runge–Kutta pair, orders 2 and 3</td>
</tr>
<tr>
<td>ode113</td>
<td>Nonstiff</td>
<td>Explicit linear multistep, orders 1 to 13</td>
</tr>
<tr>
<td>odes15s</td>
<td>Stiff</td>
<td>Implicit linear multistep, orders 1 to 5</td>
</tr>
<tr>
<td>ode23s</td>
<td>Stiff</td>
<td>Modified Rosenbrock pair (one-step), orders 2 and 3</td>
</tr>
<tr>
<td>ode23t</td>
<td>Mildly stiff</td>
<td>Trapezoidal rule (implicit), orders 2 and 3</td>
</tr>
<tr>
<td>ode23tb</td>
<td>Stiff</td>
<td>Implicit Runge–Kutta type algorithm, orders 2 and 3</td>
</tr>
</tbody>
</table>
Example 1

- **`quiver(x,y,u,v,scale)`**: plots arrows with components \((u,v)\) at the location \((x,y)\).

- The length of the arrows is \(scale\) times the norm of the \((u,v)\) vector.

To plot the blue arrows:

```matlab
%vector_field.m
n=15;
tpts = linspace(0,10,n);
ypts = linspace(0,2,n);
[t,y] = meshgrid(tpts,ypts);
pt = ones(size(y));
py = y.*(1-y);
quiver(t,y,pt,py,1);
xlim([0 10]), ylim([0 2])
```

\[
\dot{y} = y(1 - y)
\]
\[
\dot{y} = y(1 - y) \quad y(0) = 0.1
\]

To plot the solution (in red):

```matlab
tspan = [0 10];
yzero = 0.1;
[t, y] = ode45(@myf,tspan,yzero);
plot(t,y,'r--'); xlabel t; ylabel y(t)
```

function yprime = myf(t,y)
yprime = y.*(1-y);
```

The solution is always tangent to the arrows

Remember: HOLD to plot together the blue arrows & the trajectory.
\[ \dot{y} = -y - 5e^{-t}\sin 5t \quad y(0) = -0.5 \]

\begin{verbatim}
n=15; tpts = linspace(0,3,n); ypts = linspace(-1.5,1.5,n); [t,y] = meshgrid(tpts,ypts); pt = ones(size(y)); py = -y-5*exp(-t).*sin(5*t); quiver(t,y,pt,py,1); xlim([0 3.2]), ylim([-1.5 1.5])
\end{verbatim}

\begin{verbatim}
tspan = [0 3]; yzero = -0.5; [t, y] = ode45(@myf,tspan,yzero); plot(t,y,'kv--'); xlabel t; ylabel y(t)
\end{verbatim}

\begin{verbatim}
function yprime = myf(t,y) yprime = -y -5*exp(-t)*sin(5*t);
\end{verbatim}
[t,y] = ode45(@fun,tspan,yzero,options,p1,p2,...);

The optional trailing arguments p1, p2, ... represent problem parameters that, if provided, are passed on to the function fun. The optional argument options is a structure that controls many features of the solver and can be set via the odeset function. In our next example we create a structure options by the assignment

    options = odeset('AbsTol',1e-7,'RelTol',1e-4);

Passing this structure as an input argument to ode45 causes the absolute and relative error tolerances to be set to $10^{-7}$ and $10^{-4}$, respectively. (The default values are $10^{-6}$ and $10^{-3}$; see help odeset for the precise meaning of the tolerances.) These
2.8.3  (Calibrating the Euler method) The goal of this problem is to test the Euler method on the initial value problem \( \dot{x} = -x, \ x(0) = 1 \).

a) Solve the problem analytically. What is the exact value of \( x(1) \)?

b) Using the Euler method with step size \( \Delta t = 1 \), estimate \( x(1) \) numerically—call the result \( \hat{x}(1) \). Then repeat, using \( \Delta t = 10^{-n} \), for \( n = 1, 2, 3, 4 \).

c) Plot the error \( E = |\hat{x}(1) - x(1)| \) as a function of \( \Delta t \). Then plot \( \ln E \) vs. \( \ln t \). Explain the results.
2.4.9 (Critical slowing down) In statistical mechanics, the phenomenon of “critical slowing down” is a signature of a second-order phase transition. At the transition, the system relaxes to equilibrium much more slowly than usual. Here’s a mathematical version of the effect:

a) Obtain the analytical solution to $\dot{x} = -x^3$ for an arbitrary initial condition. Show that $x(t) \to 0$ as $t \to \infty$, but that the decay is not exponential. (You should find that the decay is a much slower algebraic function of $t$.)

b) To get some intuition about the slowness of the decay, make a numerically accurate plot of the solution for the initial condition $x_0 = 10$, for $0 \leq t \leq 10$. Then, on the same graph, plot the solution to $\dot{x} = -x$ for the same initial condition.

$$x(t) = \frac{x(0)}{\sqrt{1 + 2tx^2(0)}}$$
- Introduction to dynamical systems
- Introduction to flows on the line
- Solving equations with computer
- **Fixed points and linear stability**
Example

\[
\dot{x} = \sin x
\]

Analytical Solution: \[ t = \ln \left| \frac{\csc x_0 + \cot x_0}{\csc x + \cot x} \right| \]

- Starting from \( x_0 = \pi/4 \), what is the long-term behavior (what happens when \( t \to \infty \)?)

- And for any arbitrary condition \( x_0 \)?

- We look at the “phase portrait”: geometrically, picture of all possible trajectories (without solving the ODE analytically).

- Imagine: \( x \) is the position of an imaginary particle restricted to move in the line, and \( dx/dt \) is its velocity.
Imaginary particle moving in the horizontal axis

\[ \dot{x} = \sin x \]
Flow to the right when \( \dot{x} > 0 \)
Flow to the left when \( \dot{x} < 0 \)

\[ \dot{x} = 0 \] “Fixed points”

Two types of FPs: stable & unstable
Fixed points

\[ \dot{x} = f(x) \quad f(x^*) = 0 \]

\[ x = x^* \text{ initially, then } x(t) = x^* \text{ for all time} \]

Fixed points = equilibrium solutions

- **Stable** (attractor or sink): nearby trajectories are attracted
  \[ \pi \text{ and } -\pi \]

- **Unstable**: nearby trajectories are repelled
  \[ 0 \text{ and } \pm 2\pi \]
Example 1

\[ \dot{x} = x^2 - 1 \]

- Find the fixed points and classify their stability

\[ x^* = -1 \] is stable, and \[ x^* = 1 \] is unstable
Example 2

\[-V_0 + R\dot{Q} + Q/C = 0\]

\[\dot{Q} = f(Q) = \frac{V_0}{R} - \frac{Q}{RC}\]
Example 3: population model for single species (e.g., bacteria)

- $N(t)$: size of the population of the species at time $t$
  \[
  \frac{dN}{dt} = \text{births} - \text{deaths} + \text{migration}
  \]

- Simplest model (Thomas Malthus 1798): no migration, births and deaths are proportional to the size of the population
  \[
  \frac{dN}{dt} = bN - dN \quad \Rightarrow \quad N(t) = N_0e^{(b-d)t}
  \]
  Exponential growth!
More realistic model: logistic equation

- To account for limited food (Verhulst 1838):
  
  \[ \dot{N} = rN \left(1 - \frac{N}{K}\right) \]

  - If \( N > K \) the population decreases
  - If \( N < K \) the population increases

  - \( K = \text{“carrying capacity”} \)

- The carrying capacity of a biological species in an environment is the maximum population size of the species that the environment can sustain indefinitely, given the food, habitat, water, etc.
How does a population approach the carrying capacity?

- Exponential or sigmoid approach.
- Good model only for simple organisms that live in constant environments.

\[ \dot{N} = rN \left(1 - \frac{N}{K}\right) \]
Hyperbolic growth!

Technological advance
→ increase in the carrying capacity of land for people
→ demographic growth
→ more people
→ more potential inventors
→ acceleration of technological advance
→ accelerating growth of the carrying capacity...

Source: wikipedia
\[ \dot{x} = f(x) \quad f(x^*) = 0 \quad \eta(t) = x(t) - x^* \quad \eta = \text{tiny perturbation} \]

\[ \dot{\eta} = \frac{d}{dt} (x - x^*) = \dot{x} \]

\[ \dot{\eta} = \dot{x} = f(x) = f(x^* + \eta) \]

Taylor expansion

\[ f(x^* + \eta) = f(x^*) + \eta f'(x^*) + O(\eta^2) \]

\[ \dot{\eta} = \eta f'(x^*) + O(\eta^2) \]

The slope \( f'(x^*) \) at the fixed point determines the stability

\[ f'(x^*) > 0 \quad \text{the perturbation } \eta \text{ grows exponentially} \]

\[ f'(x^*) < 0 \quad \text{the perturbation } \eta \text{ decays exponentially} \]

\[ f'(x^*) = 0 \quad \text{Second-order terms can not be neglected and a nonlinear stability analysis is needed.} \]

Bifurcation (more latter)

\[ 1/|f'(x^*)| \quad \text{Characteristic time-scale} \]
Existence and uniqueness

- Problem: $f'(0)$ infinite

- When the solution of $\frac{dx}{dt} = f(x)$ with $x(0) = x_0$ exists and is unique?

- Short answer: if $f(x)$ is “well behaved”, then a solution exists and is unique.

- “well behaved”?

- $f(x)$ and $f'(x)$ are both continuous on an interval of $x$-values and that $x_0$ is a point in the interval.

- Details: see Strogatz section 2.3.

The solution to $\dot{x} = x^{1/3}$ starting from $x_0 = 0$ is not unique.

$x(t) = 0$ \hspace{1cm} x(t) = \left(\frac{2}{3} t\right)^{3/2}$
Example 1

- Linear stability of the fixed points of \( \dot{x} = \sin x \)

\[
x^* = k\pi
\]

\[
f'(x^*) = \cos k\pi = \begin{cases} 1, & k \text{ even} \\ -1, & k \text{ odd.} \end{cases}
\]

- Stable: \( \pi \) and \(-\pi\)
- Unstable: \( 0, \pm 2\pi \)
Example 2

- **Logistic equation**

\[
\dot{N} = rN \left(1 - \frac{N}{K}\right)
\]

\[N^* = 0 \text{ and } N^* = K\]

\[f'(0) = r \text{ and } f'(K) = -r\]

\[\Rightarrow N^* = 0 \text{ is unstable}\]

\[N^* = K \text{ is stable}\]

The two fixed points have the same characteristic time-scale:

\[\frac{1}{|f'(N^*)|} = \frac{1}{r}\]
The population growth of the protozoan *Paramecium* in test tubes is a typical example (Figure 1.5). Under the conditions of the experiment, the population stopped growing when there were about 552 individuals per 0.5 ml. The time points show some scatter, which is caused both by the difficulty in accurately measuring population size (only a subsample of the population is counted) and by environmental variations over time and between replicate test tubes. A linear regression of the data $N'/N$ versus $N$ gives $r = 0.99$ and $K = 552$. 

\[ \dot{N} = rN \left(1 - \frac{N}{K}\right) \]
Lack of oscillations

\[ \dot{x} = f(x) \]

General observation: only sigmoidal or exponential behavior, the approach is monotonic, **no oscillations**

Analogy:

\[ m\ddot{x} + b\dot{x} = F(x) \]

Strong damping
(over damped limit)

To observe oscillations we need to keep the second derivative (weak damping).
Stability of the fixed point $x^*$ when $f'(x^*)=0$?

(a) $\dot{x} = -x^3$  ,  (b) $\dot{x} = x^3$  ,  (c) $\dot{x} = x^2$  ,  (d) $\dot{x} = 0$

In all these systems: $x^* = 0$ with $f'(x^*) = 0$

When $f'(x^*) = 0$ nothing can be concluded from the linearization but these plots allow to see what goes on.
\[
\dot{x} = f(x) \quad f(x) = -\frac{dV}{dx}
\]

\[
\frac{dV}{dt} = \frac{dV}{dx} \frac{dx}{dt} \quad \frac{dV}{dt} = -\left(\frac{dV}{dx}\right)^2 \leq 0
\]

\(V(t)\) decreases along the trajectory.

- Example: \(\dot{x} = x - x^3\)

\[V = -\frac{1}{2}x^2 + \frac{1}{4}x^4 + C\]

Two fixed points: \(x=1\) and \(x=-1\) (Bistability).
Flows on the line = first-order ODE: $dx/dt = f(x)$

Fixed point solutions: $f(x^*) = 0$
- stable if $f'(x^*) < 0$
- unstable if $f'(x^*) > 0$
- neutral (bifurcation point) if $f'(x^*) = 0$

There are no periodic solutions; the approach to a fixed point is monotonic (sigmoidal or exponential).
Steven H. Strogatz: *Nonlinear dynamics and chaos, with applications to physics, biology, chemistry and engineering.*  
First or second ed., Chapters 1 and 2