Nonlinear systems, chaos and control in Engineering

Bifurcations: saddle-node, transcritical and pitchfork

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Reminder Summary Part 1

- Flows on the line = first-order ordinary differential equations.

  $dx/dt = f(x)$

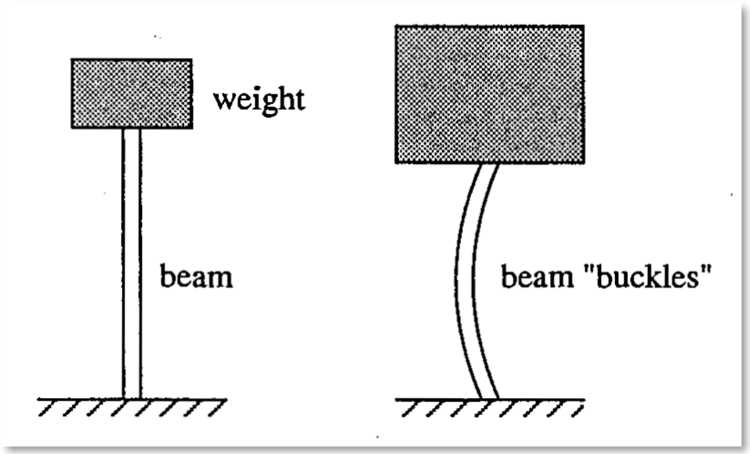
- Fixed point solutions: $f(x^*) = 0$
  - stable if $f'(x^*) < 0$
  - unstable if $f'(x^*) > 0$
  - neutral (bifurcation point) if $f'(x^*) = 0$
- There are no periodic solutions; the approach to fixed point solutions is monotonic (sigmoidal or exponential).

Outline

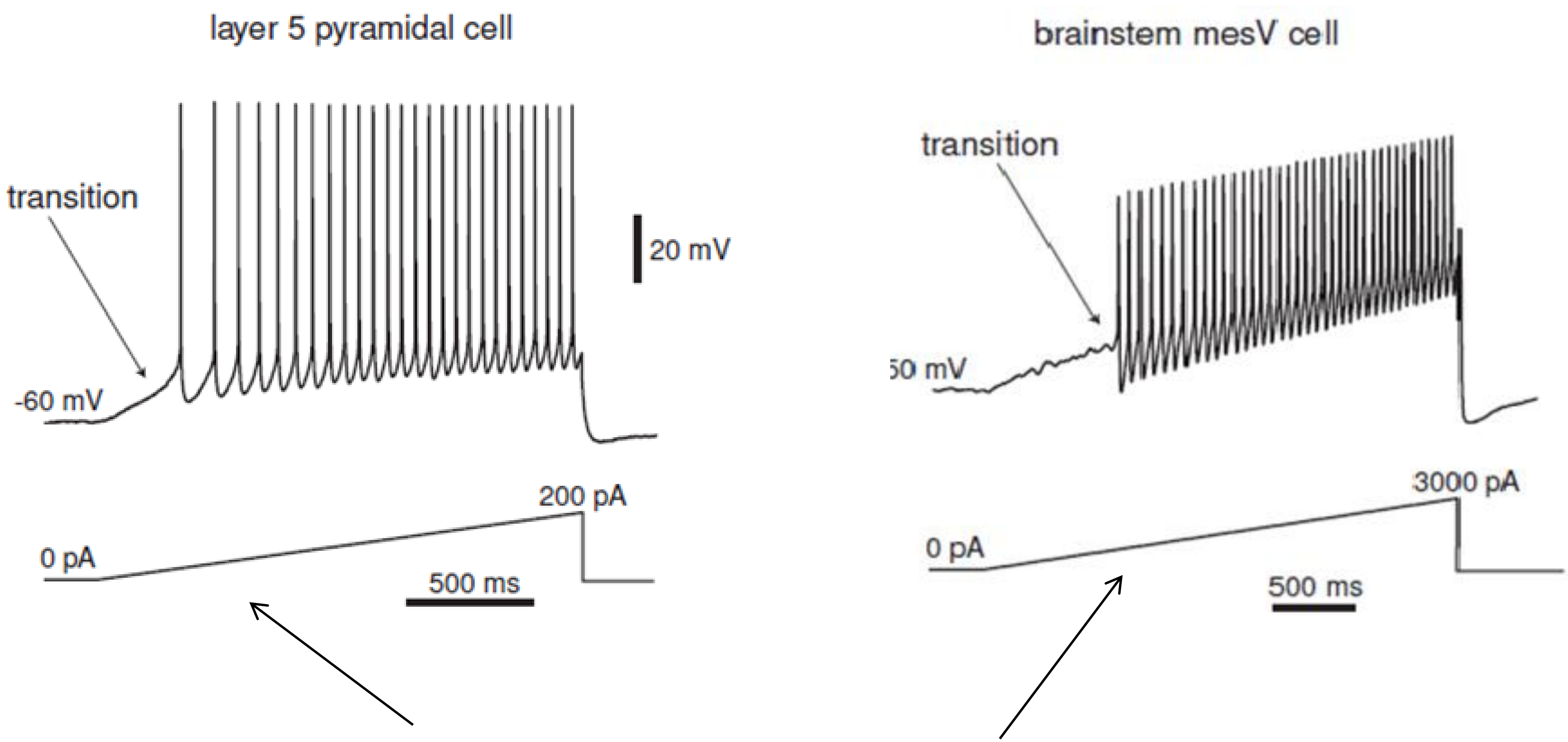
- Introduction to bifurcations
- Saddle-node, transcritical and pitchfork bifurcations
- Examples
- Imperfect bifurcations & catastrophes

What is a bifurcation?

- A qualitative change (in the structure of the phase space) when a **control parameter is varied**:
  - Fixed points can be created or destroyed
  - The stability of a fixed point can change
- There are many examples in physical systems, biological systems, etc.

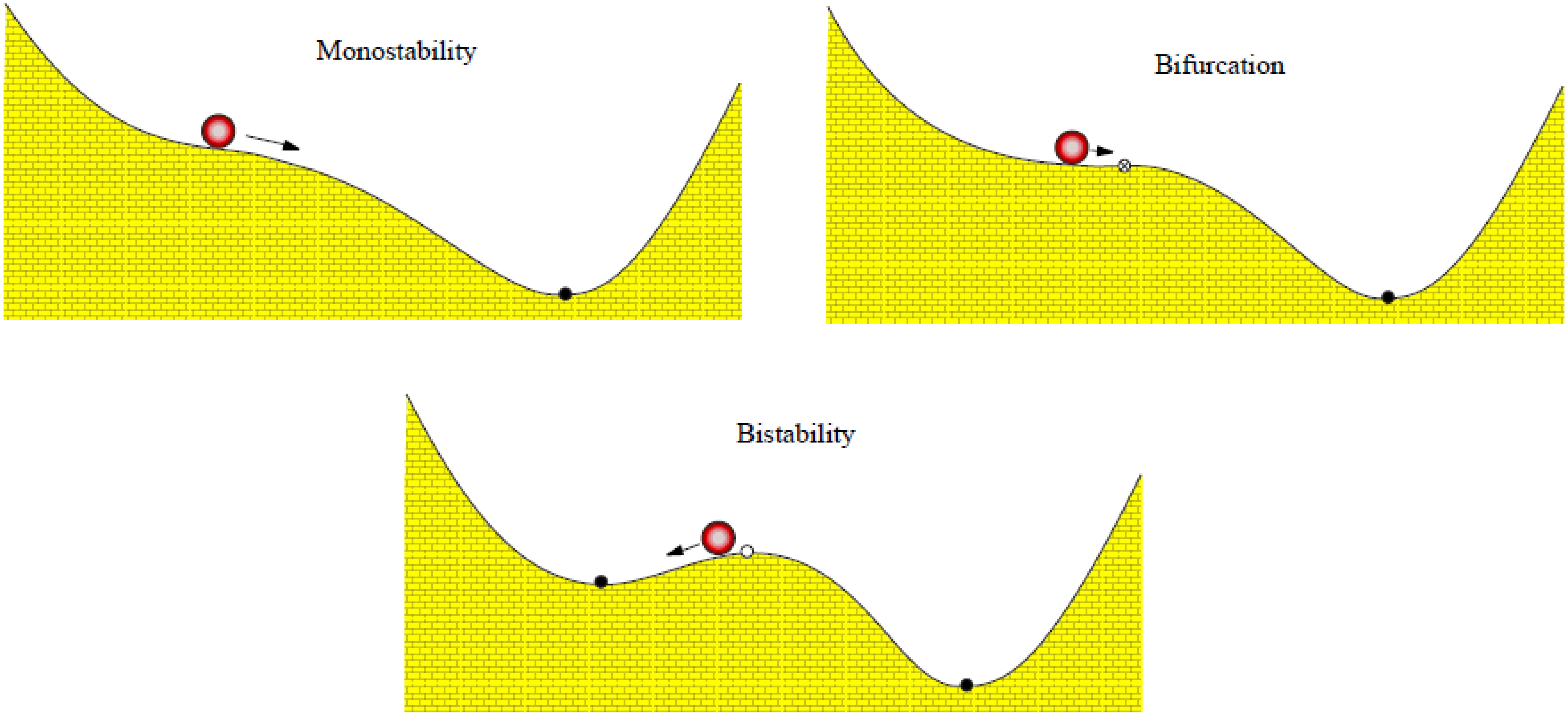


Example



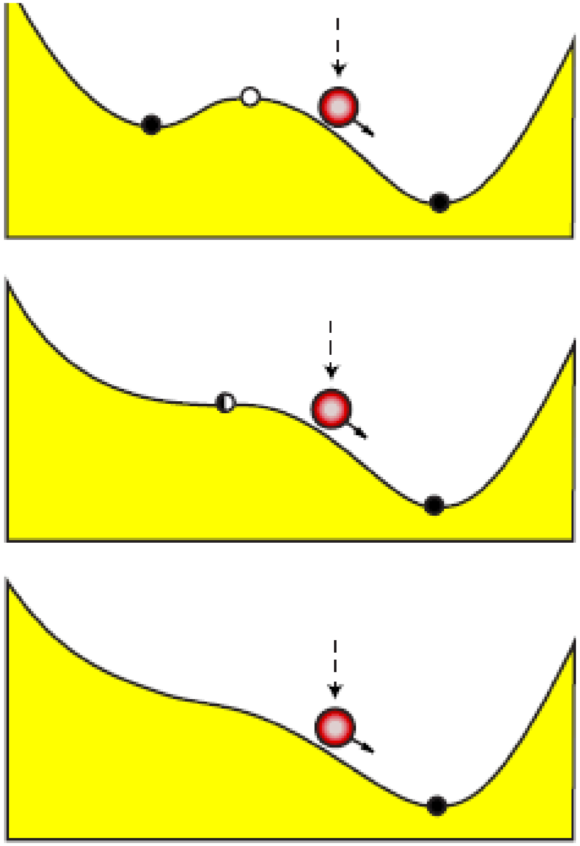
Control parameter increases in time

Bifurcation and potential

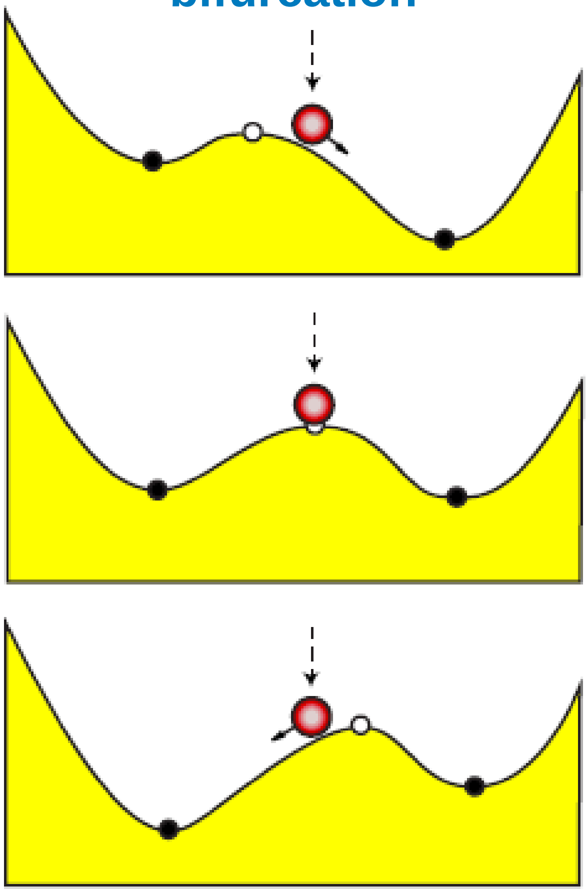


Bifurcations are not equivalent to qualitative change of behavior

Bifurcation but no change of behavior



Change of behavior but no bifurcation



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- Saddle-node, transcritical and pitchfork bifurcations
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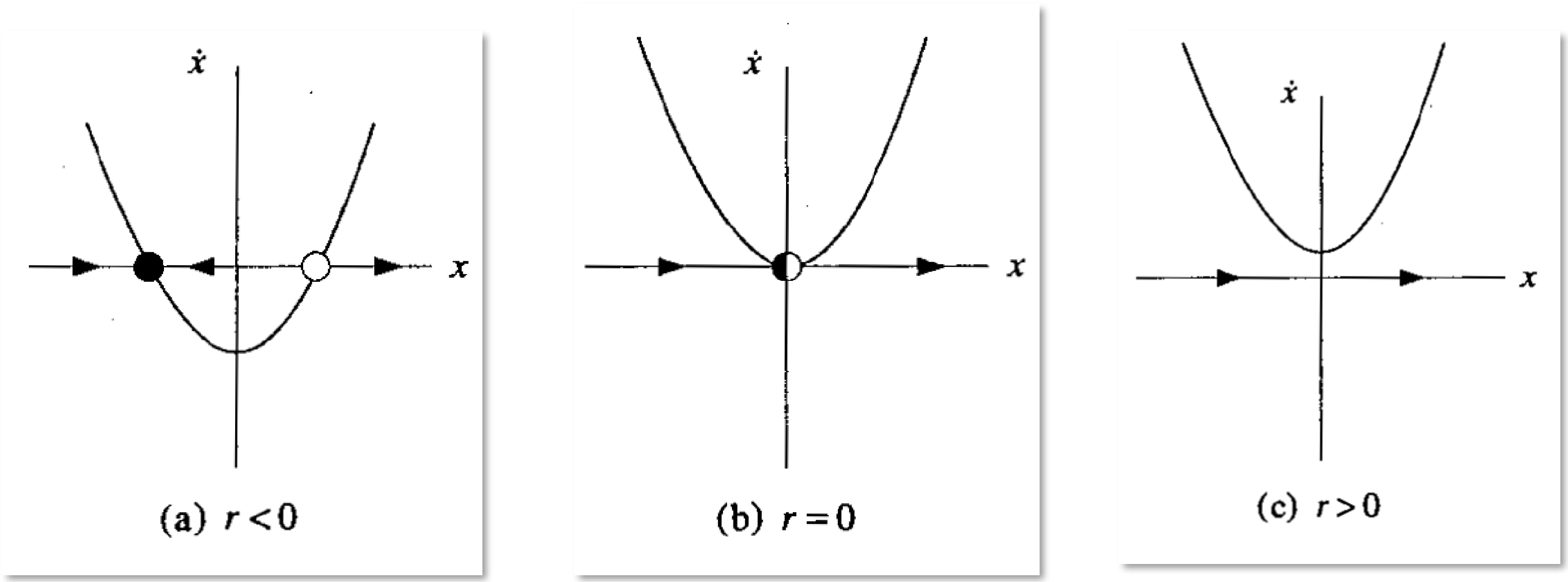
Saddle-node bifurcation

Basic mechanism for the creation or the destruction of fixed points

$$\dot{x} = f(x) = r + x^2 \qquad x^* = \pm\sqrt{-r}$$



$$f'(-\sqrt{-r}) = -2\sqrt{-r}$$ Stable if r<0

$$f'(\sqrt{-r}) = 2\sqrt{-r}$$ unstable

At the bifurcation point $r^*=0$: $f'(x^*) = 0$

Example

$$\dot{x} = r - x^2$$

- Calculate the fixed points and their stability as a function of the control parameter *r*

$$x^* = \pm\sqrt{r}$$

$r < 0$ $\qquad$ $r = 0$ $\qquad$ $r > 0$

Normal forms

- Are representative of all saddle-node bifurcations.

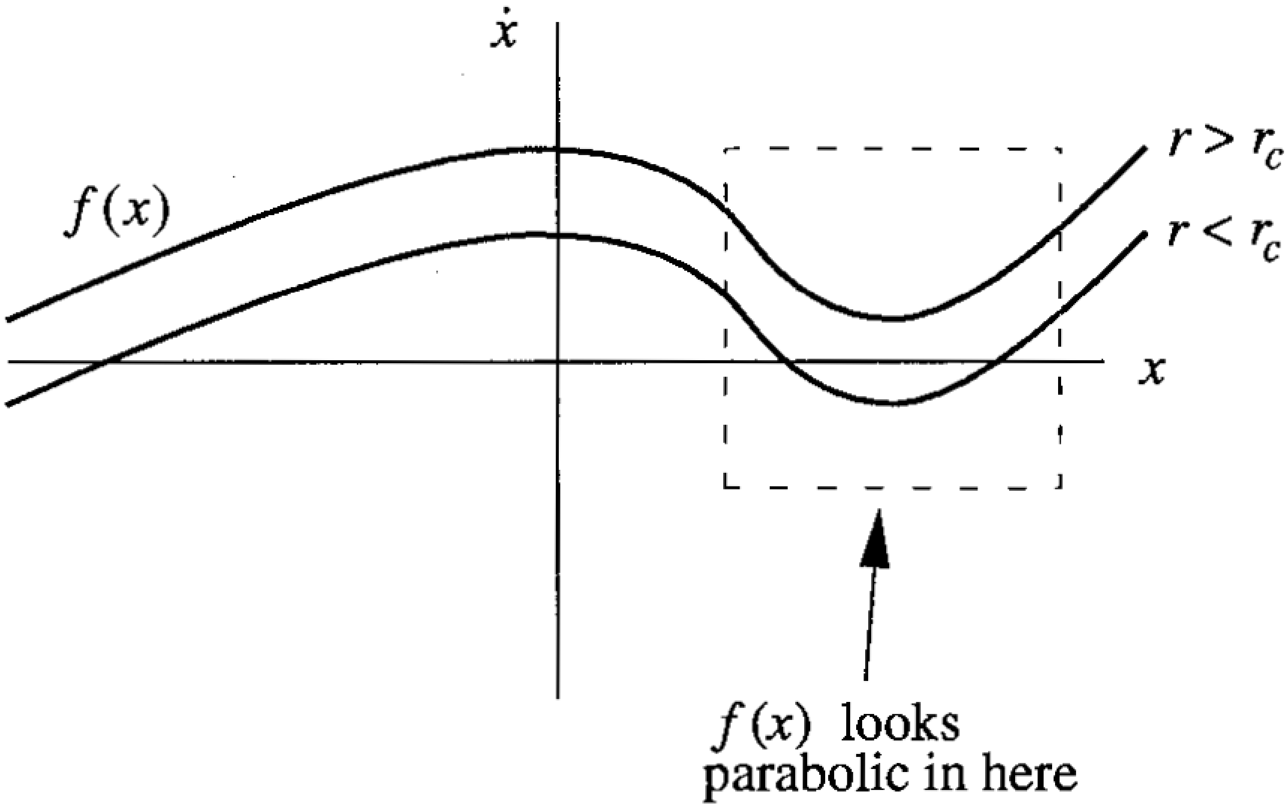
- Close to the saddle-node bifurcation the dynamics can be approximated by

$$\dot{x} = r - x^2 \quad \text{or} \quad \dot{x} = r + x^2$$

Example:

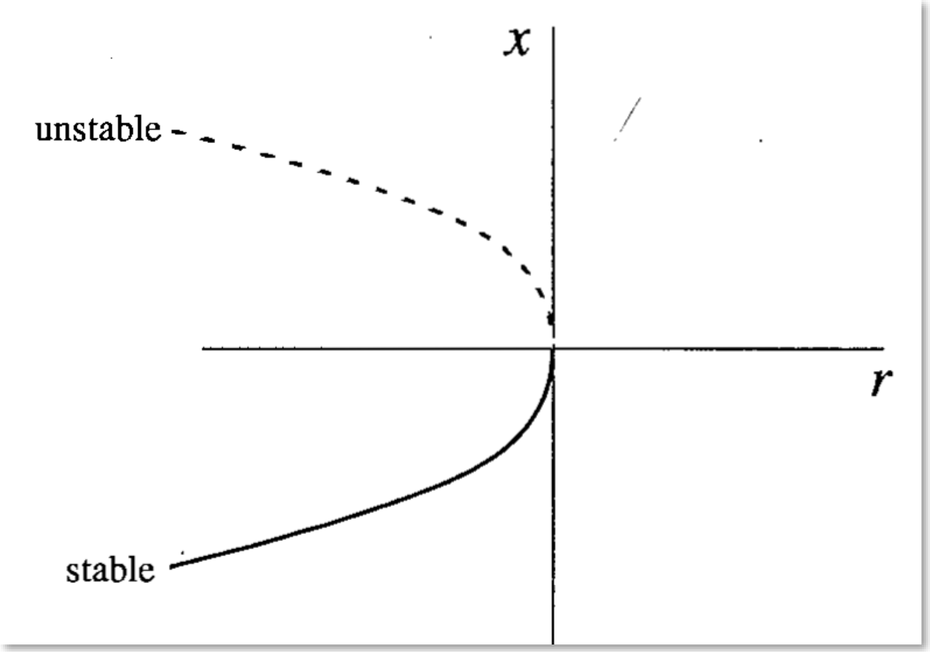
$$\begin{aligned}\dot{x} &= r - x - e^{-x} \\ &= r - x - \left[1 - x + \frac{x^2}{2!} + \cdots\right] \\ &= (r-1) - \frac{x^2}{2} + \cdots\end{aligned}$$

Near a saddle-node bifurcation



Bifurcation diagram

$$\dot{x} = r + x^2$$



Two fixed points → one fixed point → 0 fixed point

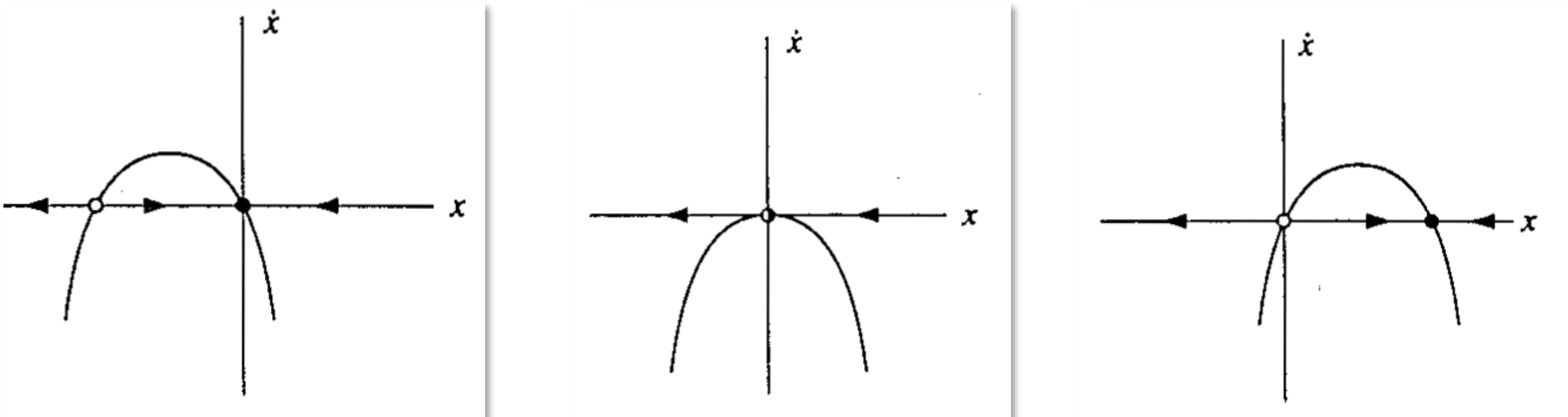
A pair of fixed points appear (or disappear) out of the "clear blue sky" ("blue sky" bifurcation, Abraham and Shaw 1988).

Transcritical bifurcation

$$\dot{x} = rx - x^2$$

$$x^* = 0$$
$$x^* = r$$

are the fixed points for all r



(a) $r < 0$ (b) $r = 0$ (c) $r > 0$

Transcritical bifurcation: general mechanism for changing the stability of fixed points.

Bifurcation diagram

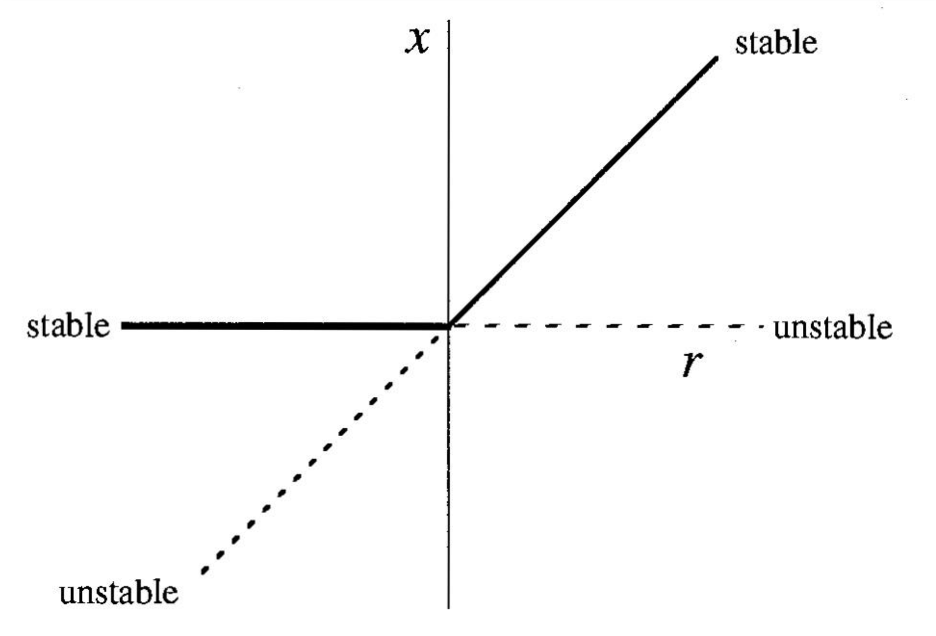
$$\dot{x} = rx - x^2$$

fixed points $x^* = 0$ and $x^* = r$

$$f'(x) = r - 2x$$

$$f'(0) = r$$

$$f'(r) = -r$$



- Exchange of stability at $r = 0$.
- **Exercise**: $\dot{x} = r \ln x + x - 1$

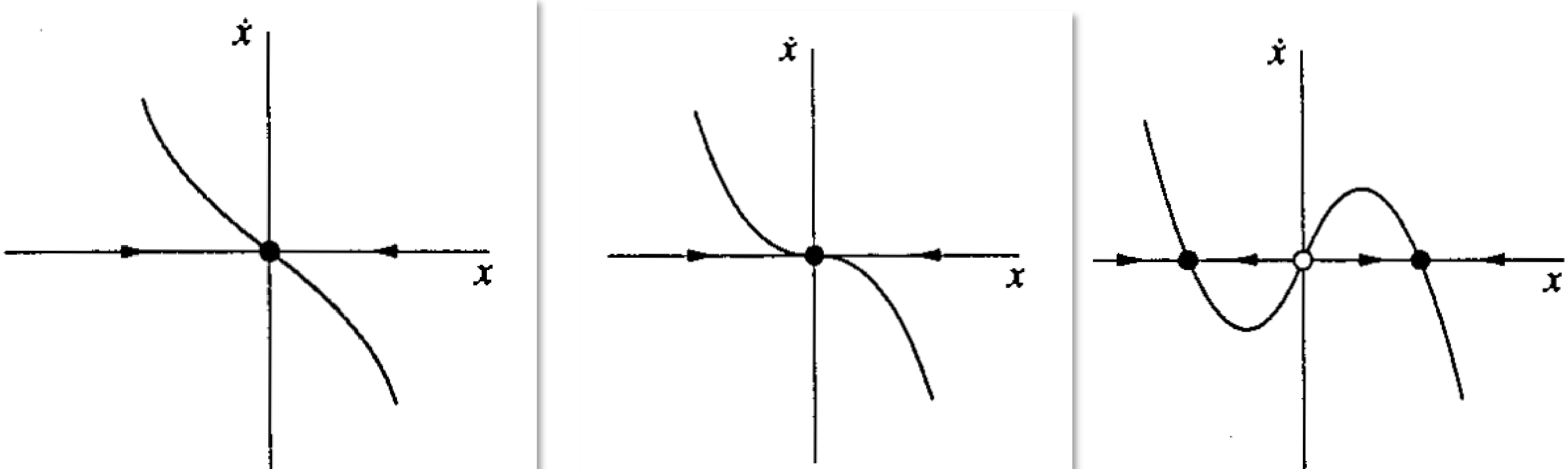
  show that a transcritical bifurcation occurs near x=1

  (hint: consider $u = x\text{-}1$ small)

Pitchfork bifurcation

$$\dot{x} = rx - x^3$$

Symmetry x → -x



(a) $r < 0$ (b) $r = 0$ (c) $r > 0$

One fixed point → 3 fixed points

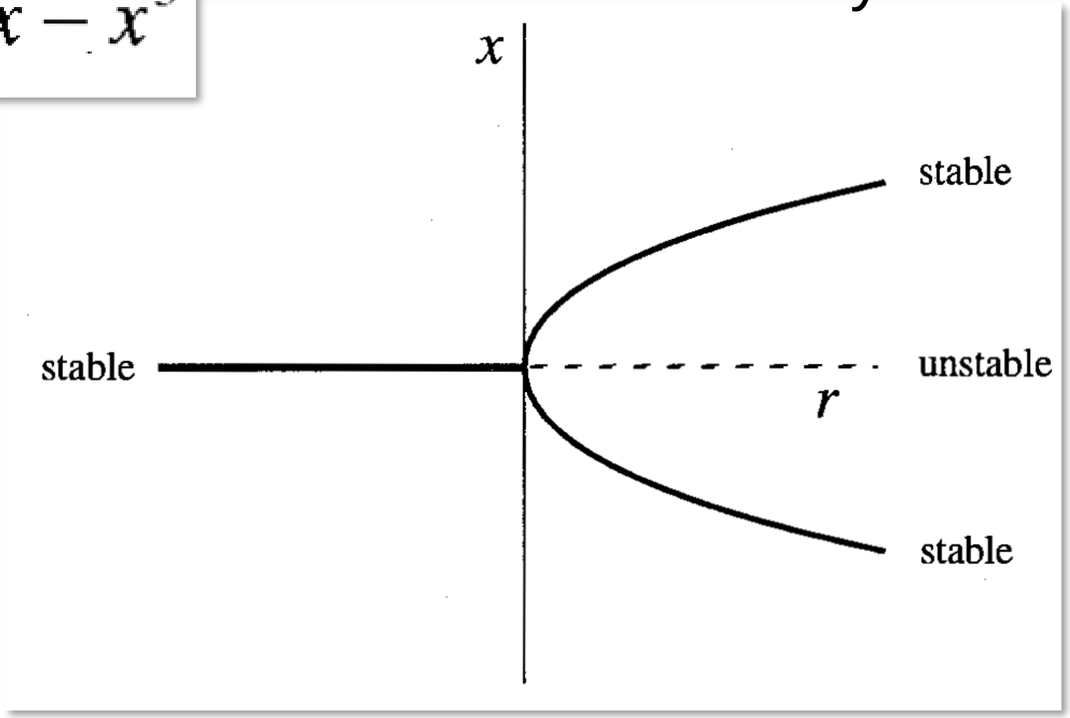
Bifurcation diagram

$$\dot{x} = rx - x^3$$

Bistability

$$x^* = 0$$

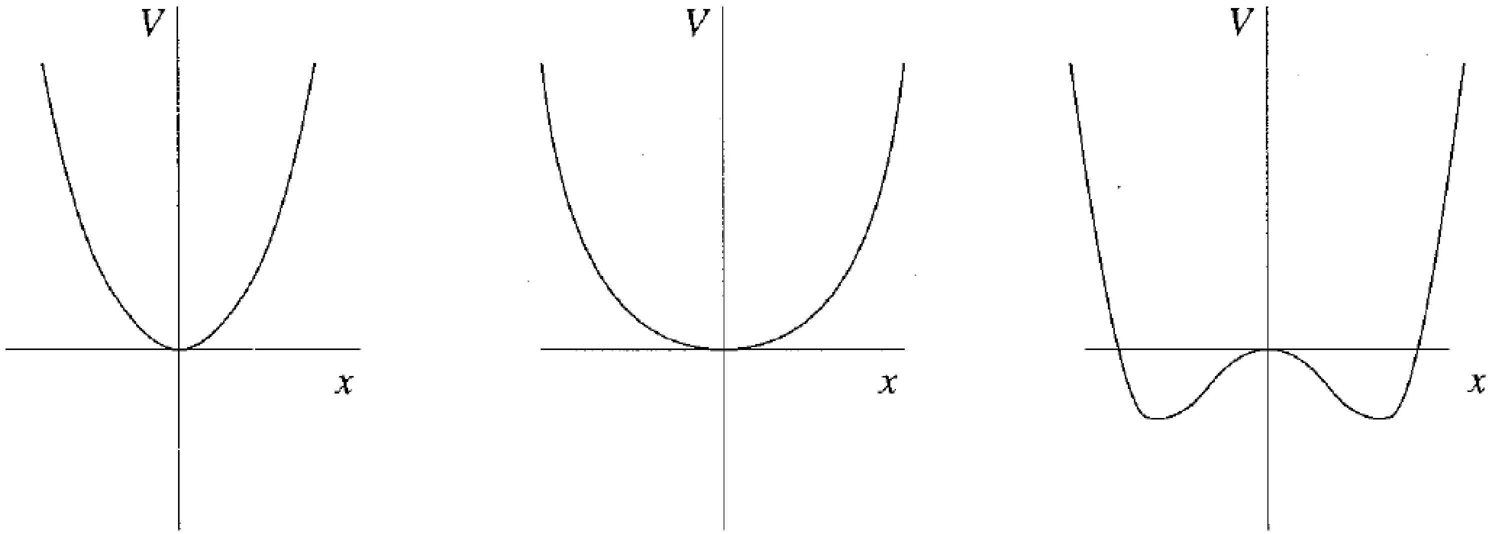
$$x^* = \pm\sqrt{-r}$$



- The governing equation is symmetric: $x \to -x$ but for $r > 0$: symmetry broken solutions.

Potential

$\dot{x} = rx - x^3$ $\qquad$ $\dot{x} = f(x)$ $\qquad$ $f(x) = -dV/dx$ $\qquad$ $V(x) = -\frac{1}{2}rx^2 + \frac{1}{4}x^4$



$r < 0$ $\qquad$ $r = 0$ $\qquad$ $r > 0$

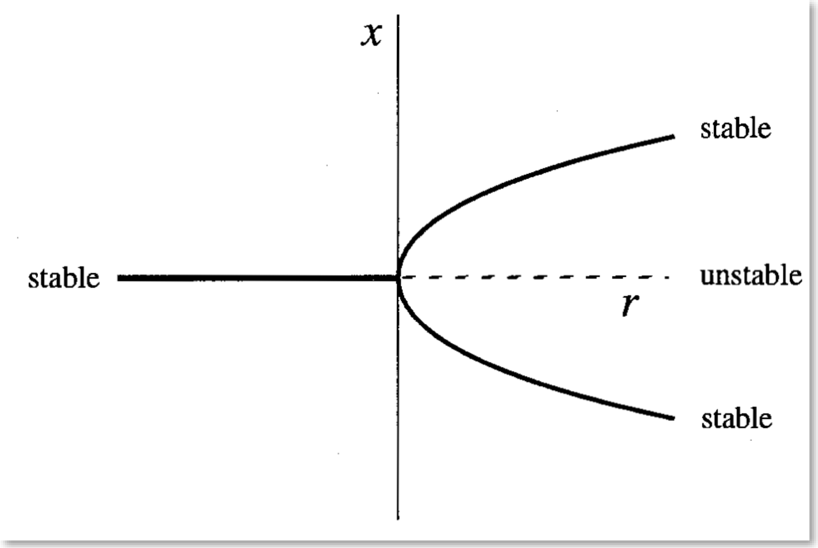
Pitchfork bifurcations

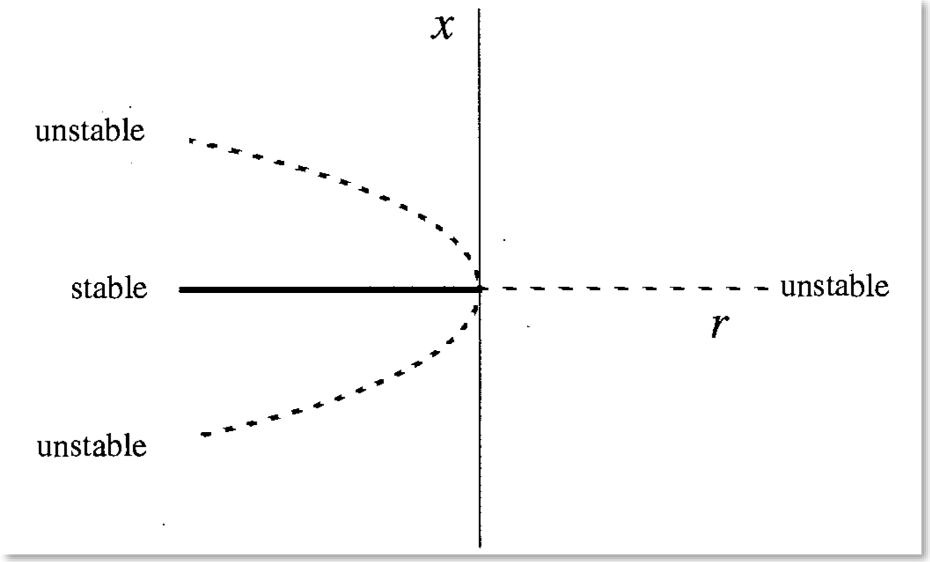
Supercritical:
x^3 is stabilizing

$$\dot{x} = rx - x^3$$

Subcritical:
x^3 is destabilizing

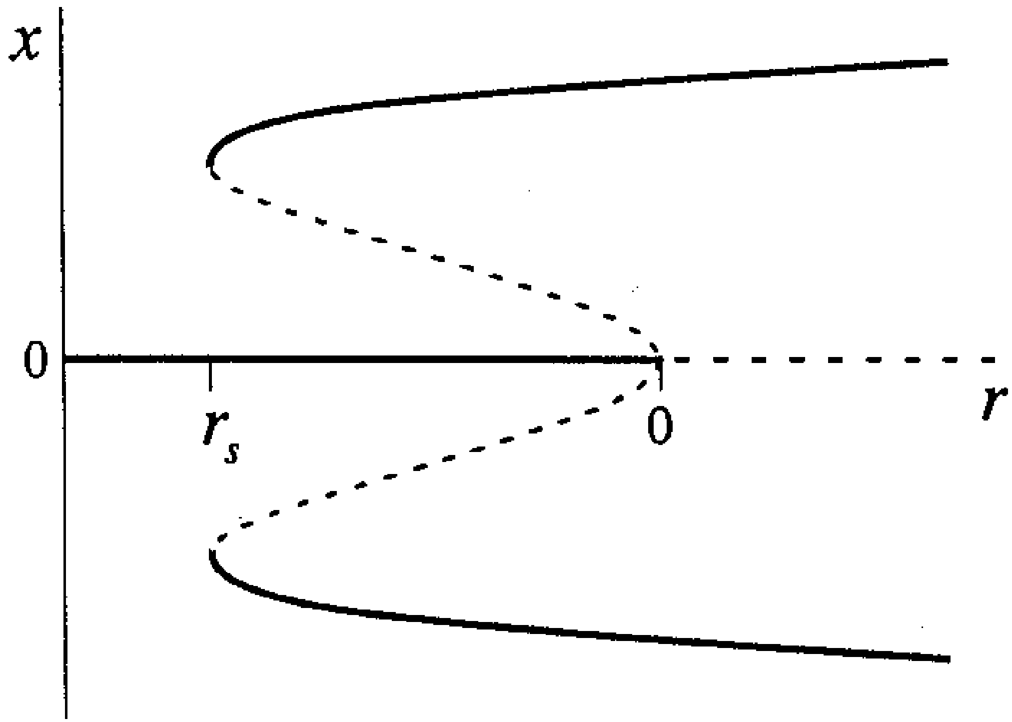
$$\dot{x} = rx + x^3$$



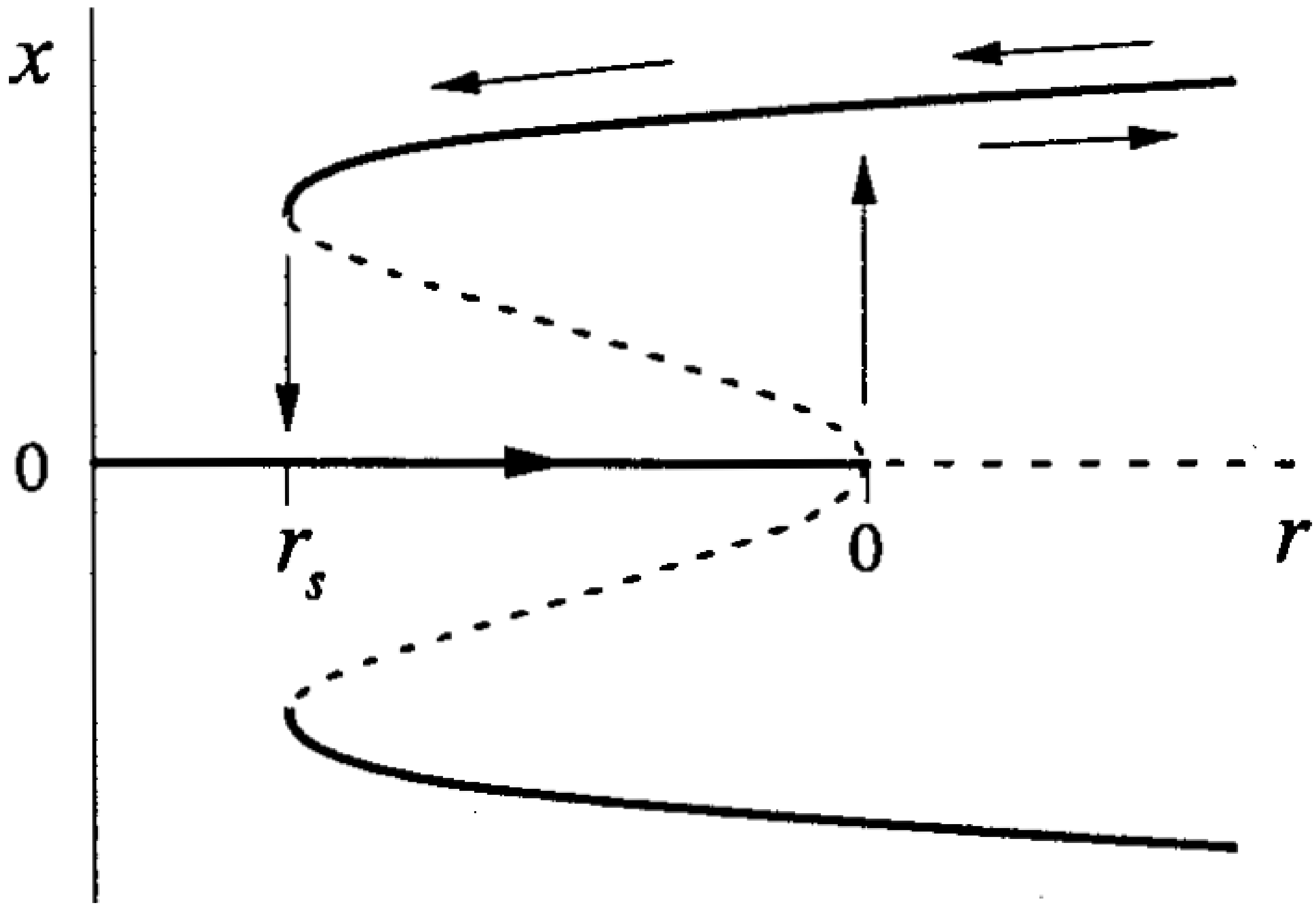


Exercise: find the fixed points and compute their stability

$$\dot{x} = rx + x^3 - x^5$$



Subcritical bifurcation: Hysteresis



Critical or dangerous transition! A lot of effort in trying to find “early warning signals” (more latter)

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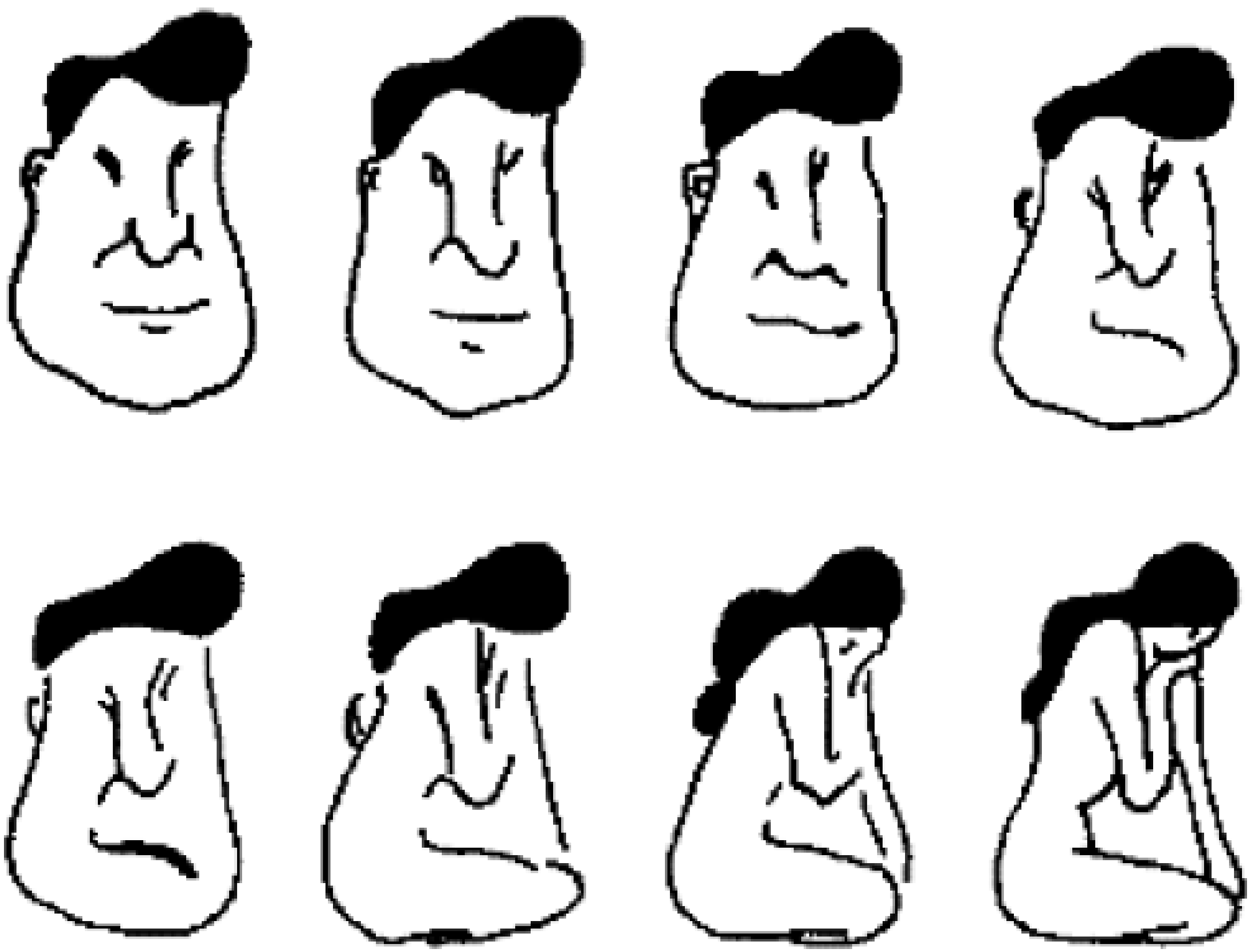
Hysteresis: sudden changes in visual perception



Fischer (1967): experiment with 57 students.

"When do you notice an abrupt change in perception?"

Summary

- Bifurcation condition: change in the stability of a fixed point

$$f\,'(x^*) = 0$$

- In first-order ODEs: three possible bifurcations
  - Saddle node
  - Pitchfork
  - Trans-critical

- The normal form describes the behavior near the bifurcation.

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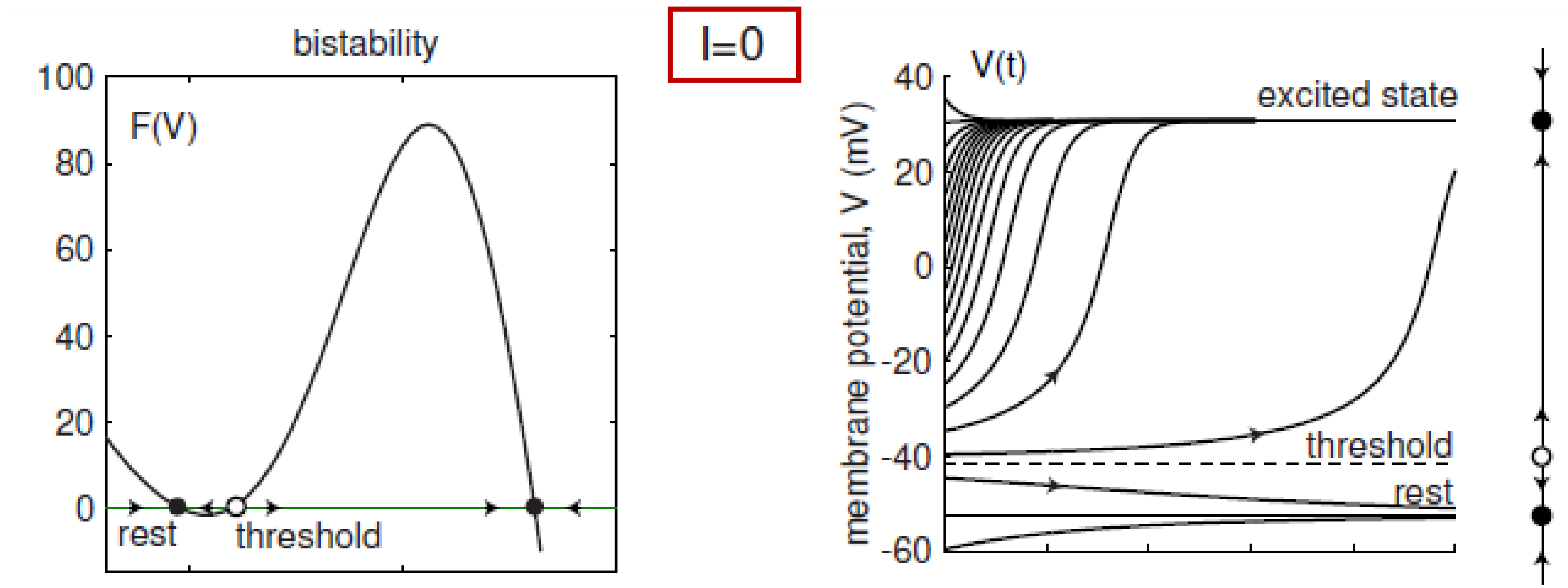
Example: neuron model

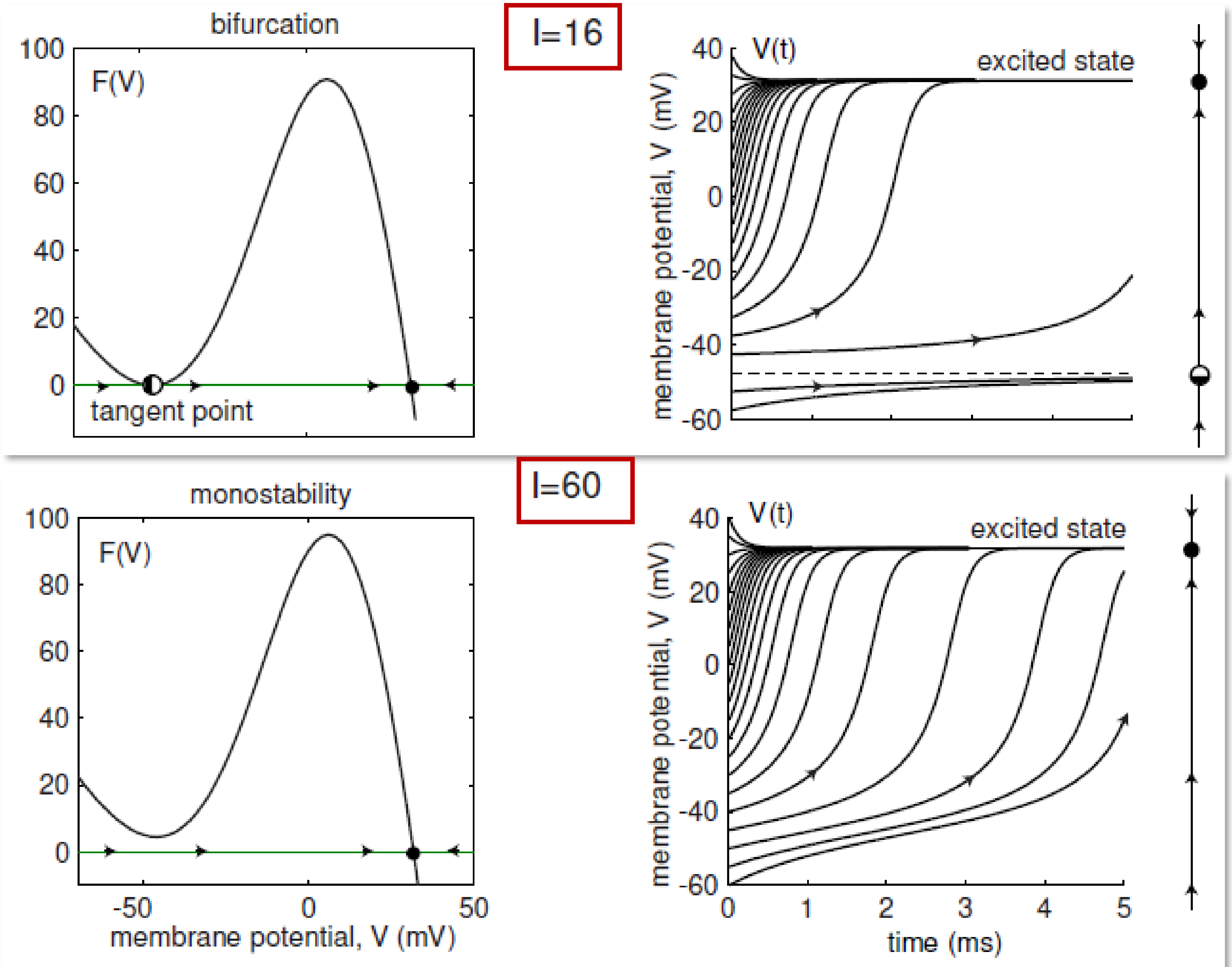
$$C\,\dot{V} = I - g_{\mathrm{L}}(V - E_{\mathrm{L}}) - \overbrace{g_{\mathrm{Na}}\,m_{\infty}(V)\,(V - E_{\mathrm{Na}})}^{\text{instantaneous } I_{\mathrm{Na,p}}}$$

$$m_{\infty}(V) = 1/(1 + \exp\{(V_{1/2} - V)/k\})$$

$$C = 10\ \mu\mathrm{F}, \quad I = 0\ \mathrm{pA}, \quad g_{\mathrm{L}} = 19\ \mathrm{mS}, \quad E_{\mathrm{L}} = -67\ \mathrm{mV},$$
$$g_{\mathrm{Na}} = 74\ \mathrm{mS}, \quad V_{1/2} = 1.5\ \mathrm{mV}, \quad k = 16\ \mathrm{mV}, \quad E_{\mathrm{Na}} = 60\ \mathrm{mV}$$

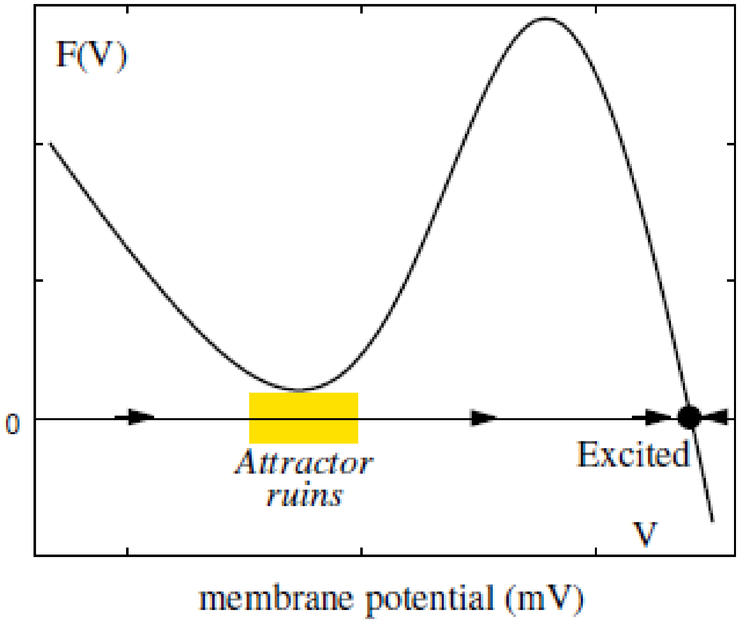


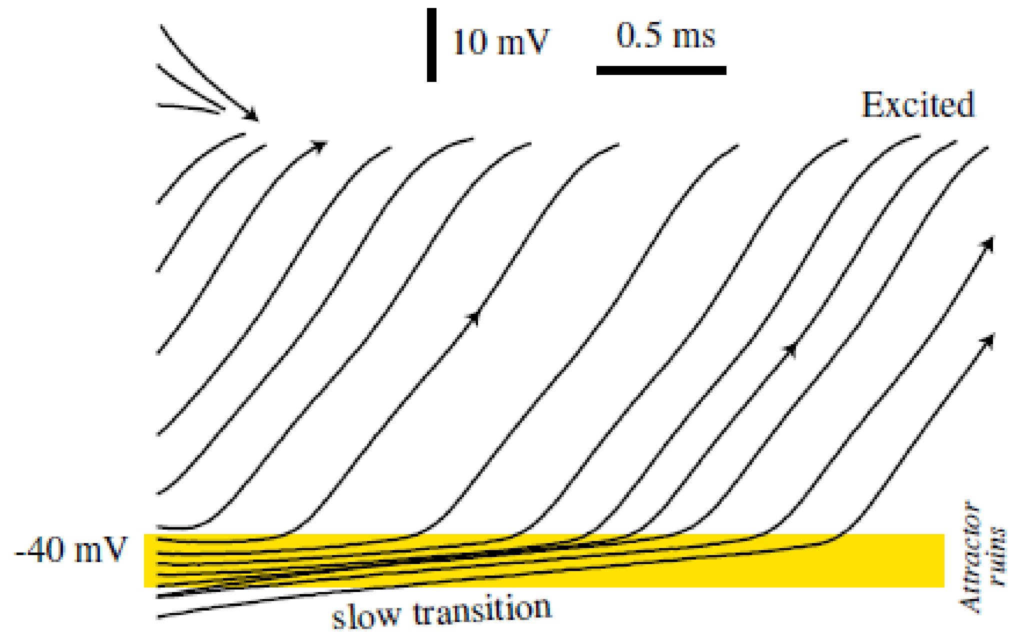
Saddle-node Bifurcation



Near the bifurcation point: slow dynamics

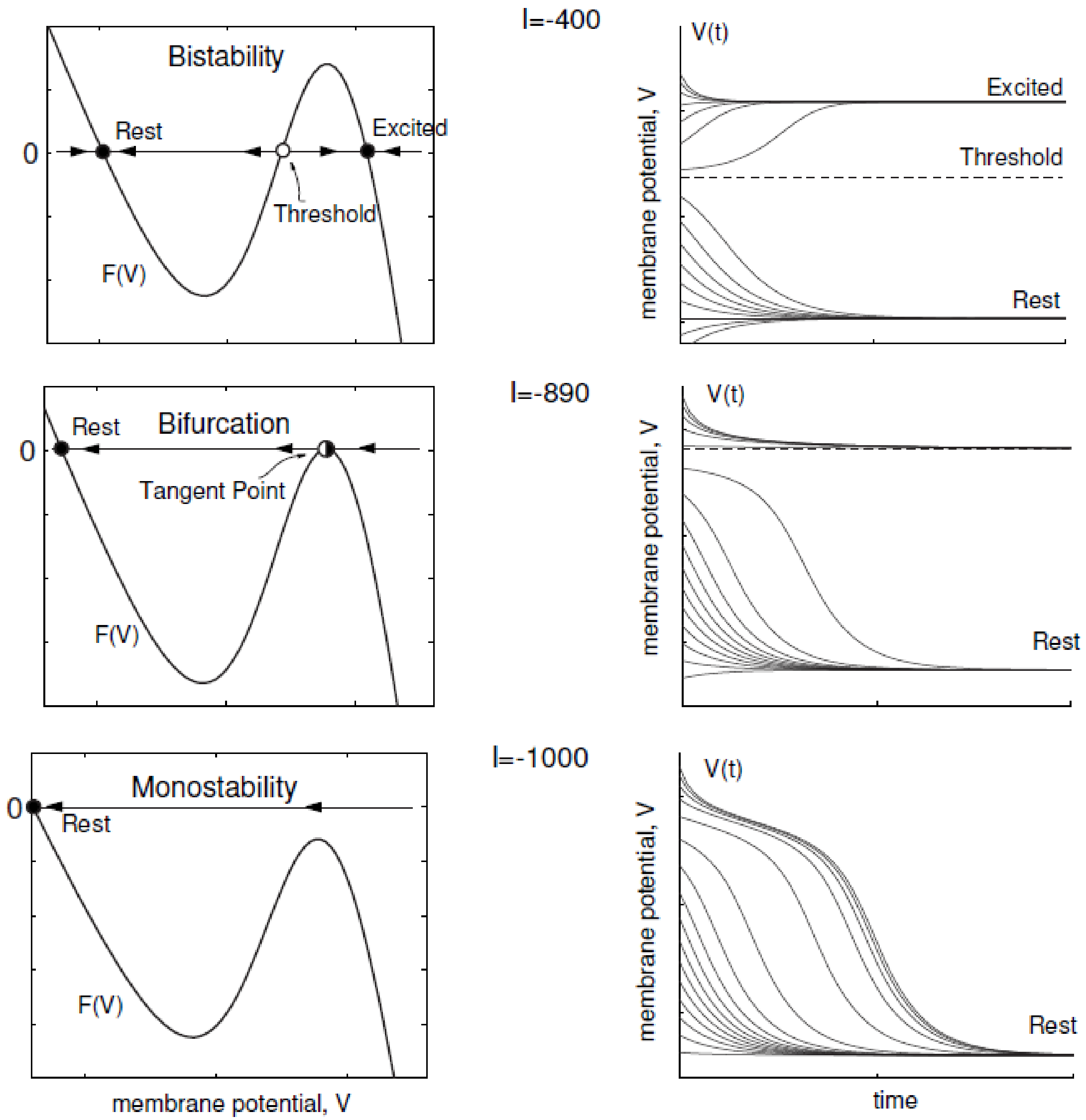
$$\boxed{I = 30 \text{ pA}}$$





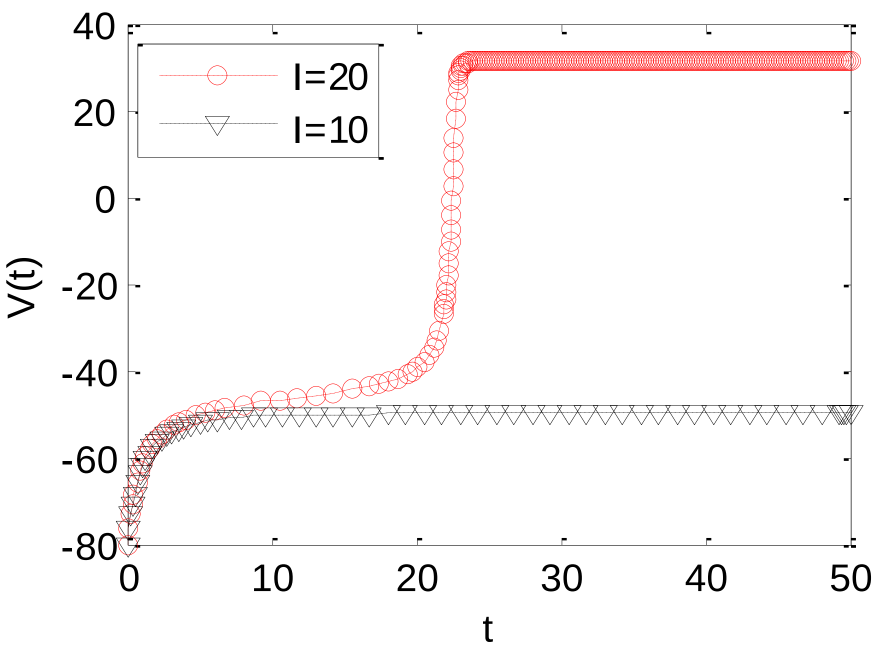
This slow transition is an "early warning signal" of a critical or dangerous transition ahead (more latter)

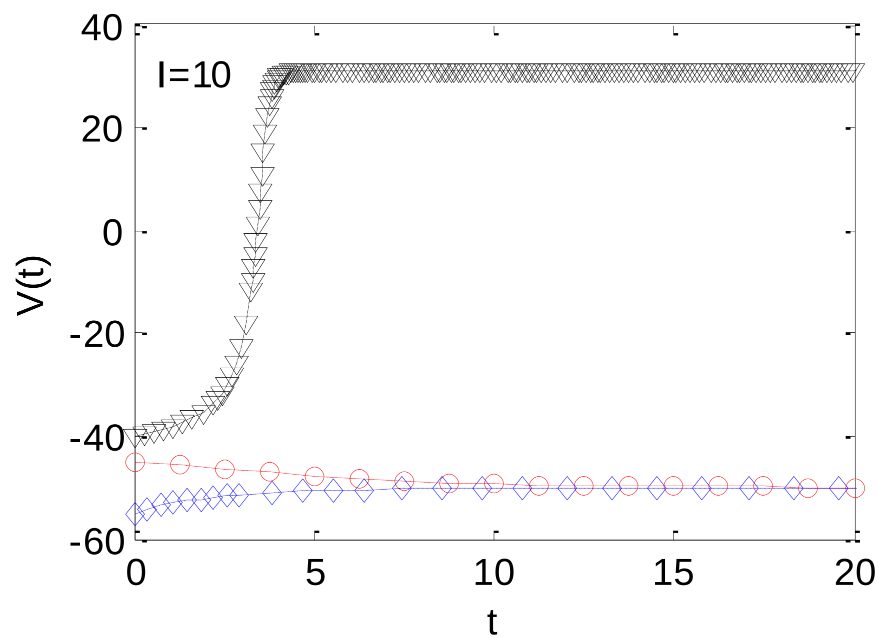
If the control parameter now decreases



Class/home work

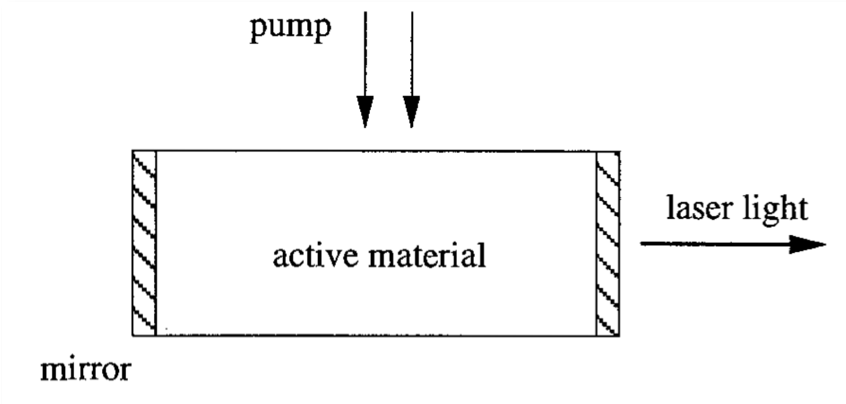
- Simulate the neuron model with different values of the control parameter I and/or different initial conditions.





Example: laser threshold

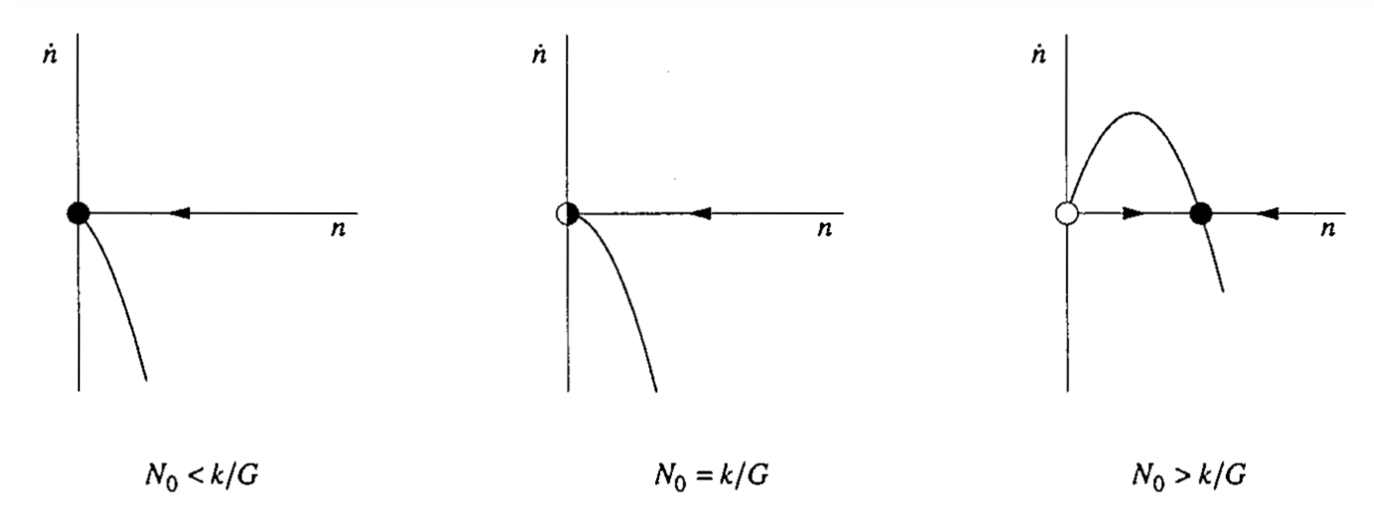
$$\dot{n} = \text{gain} - \text{loss}$$
$$= GnN - kn.$$



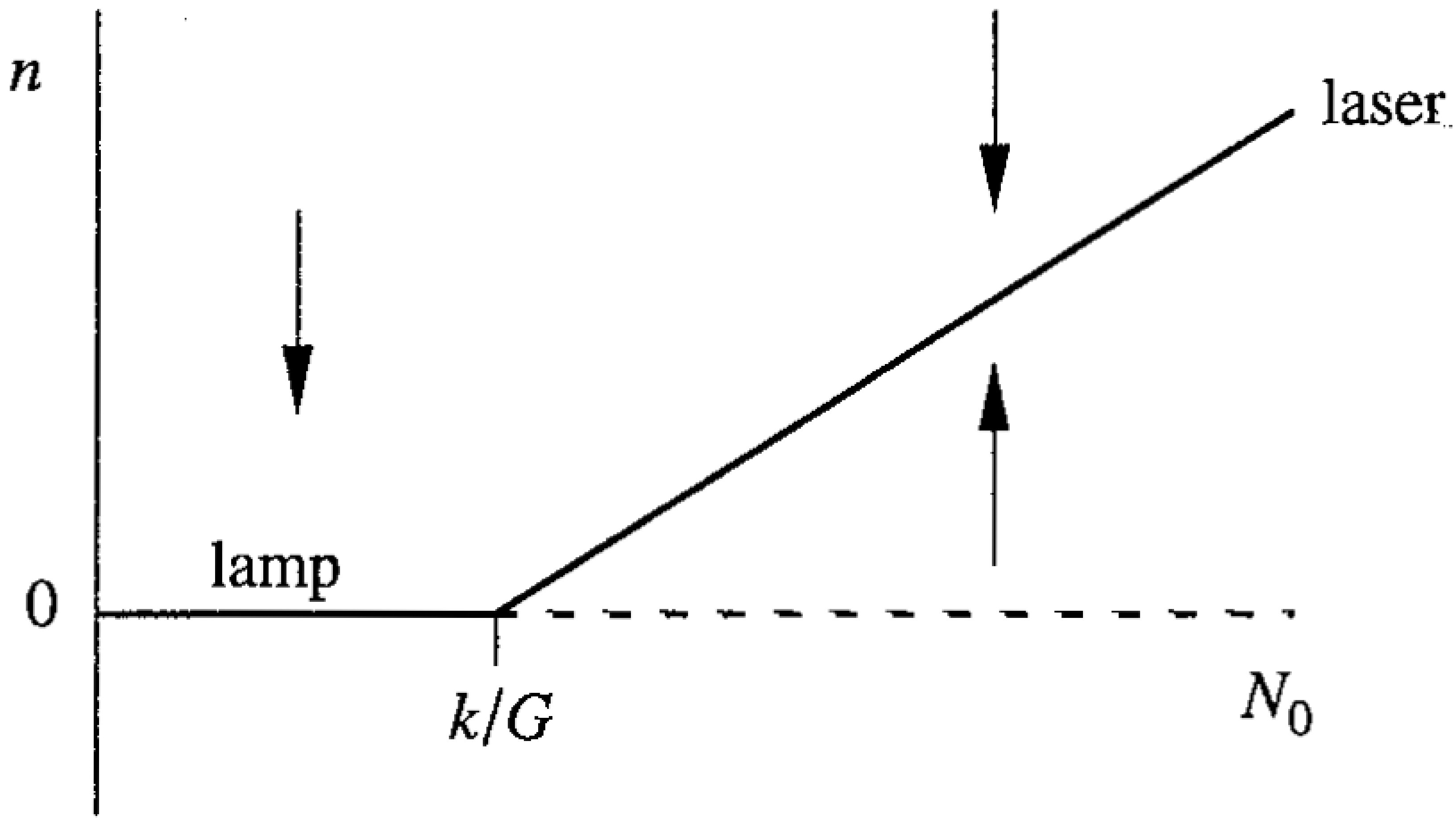
$$N(t) = N_0 - \alpha n$$

$$\dot{n} = Gn(N_0 - \alpha n) - kn$$
$$= (GN_0 - k)n - (\alpha G)n^2.$$

$$\dot{x} = rx - x^2$$



Transcritical Bifurcation



"imperfect" bifurcation due to noise

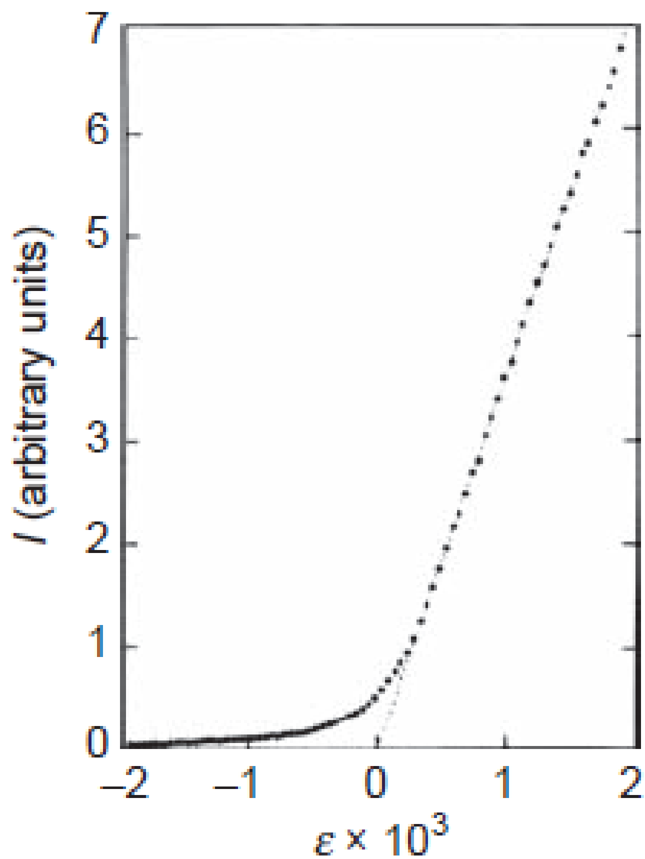


Fig. 1.17 Imperfect bifurcation for a laser in the presence of spontaneous emission, measured for a He-Ne laser. Reprinted Figure 1 with permission from Corti and Degiorgio [42]. Copyright 1976 by the American Physical Society.

Laser turn on

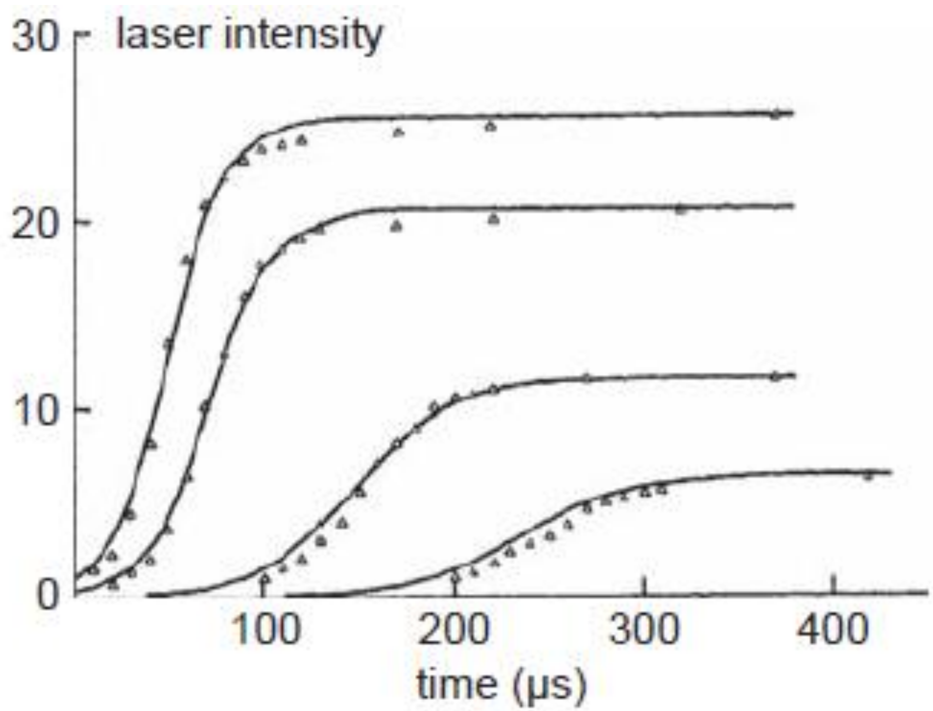


Fig. 1.3 He-Ne gas laser output as a function of time. From the lower to the upper time traces, the pump parameter above threshold is gradually increased. Reprinted Figure 2 with permission from Pariser and Marshall [30]. Copyright 1965 by the American Institute of Physics.

Laser turn-on delay

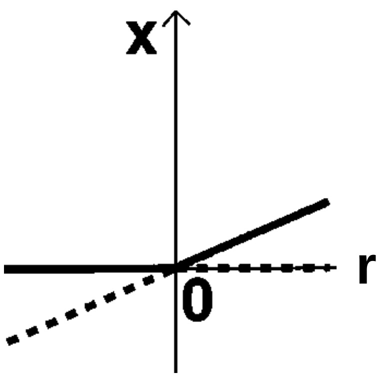
$$\dot{x} = rx - x^2$$

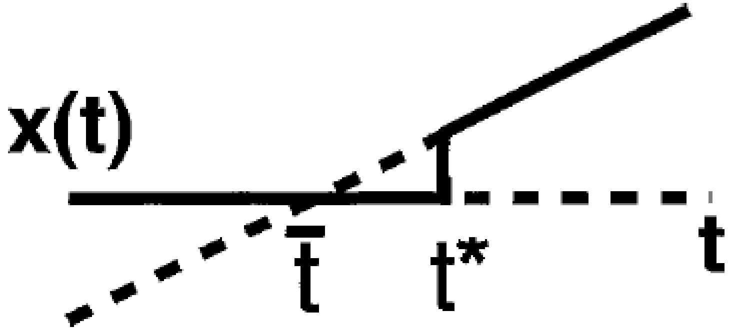
$$r(t) = r_0 + vt$$

Linear increase of control parameter

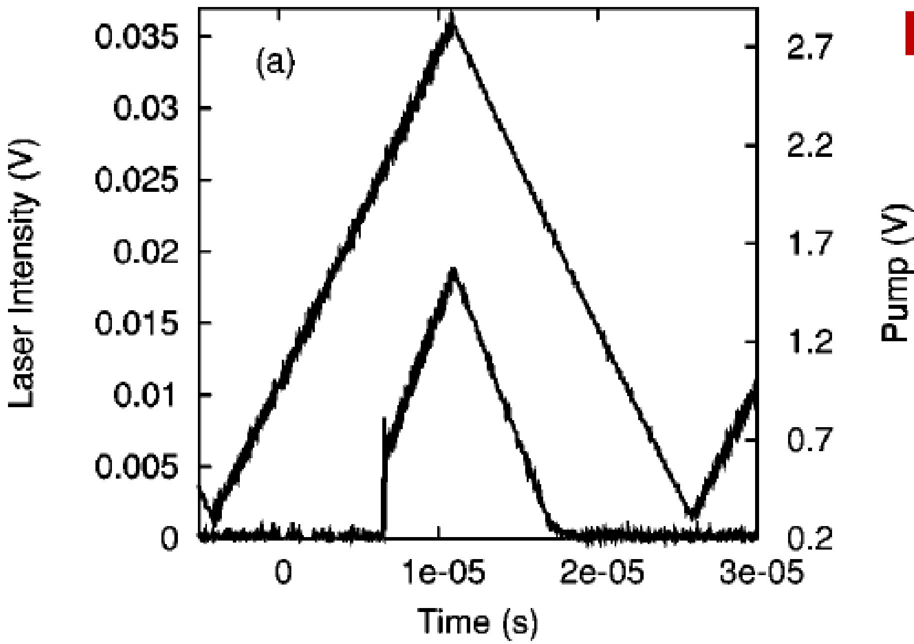
$$r_0 < r^* = 0$$

Start before the bifurcation point

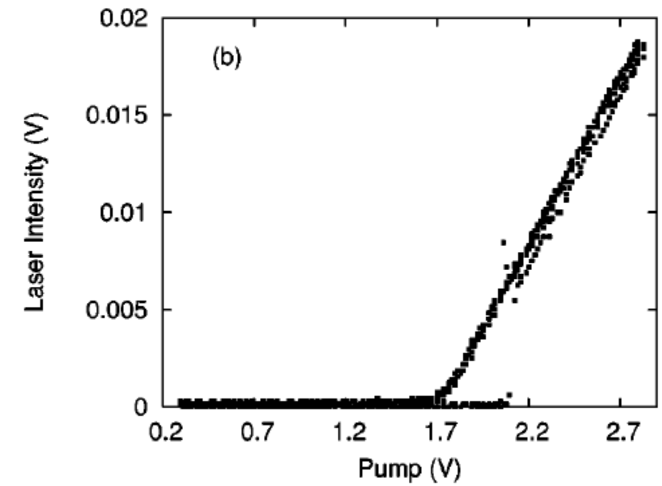




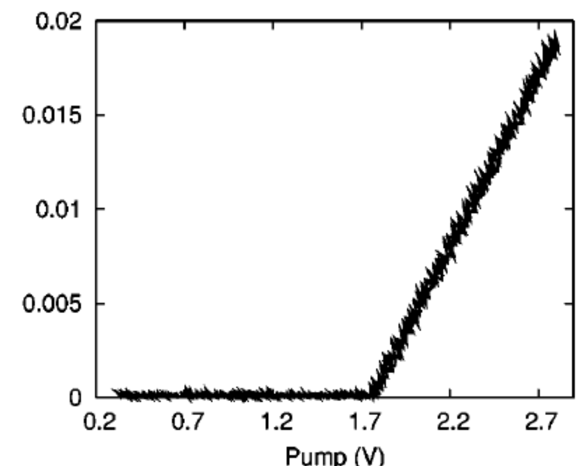
Comparison with experimental observations



Dynamical hysteresis



Quasi-static very slow variation of the control parameter



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Class/home work with matlab

$$\dot{x} = rx - x^2$$

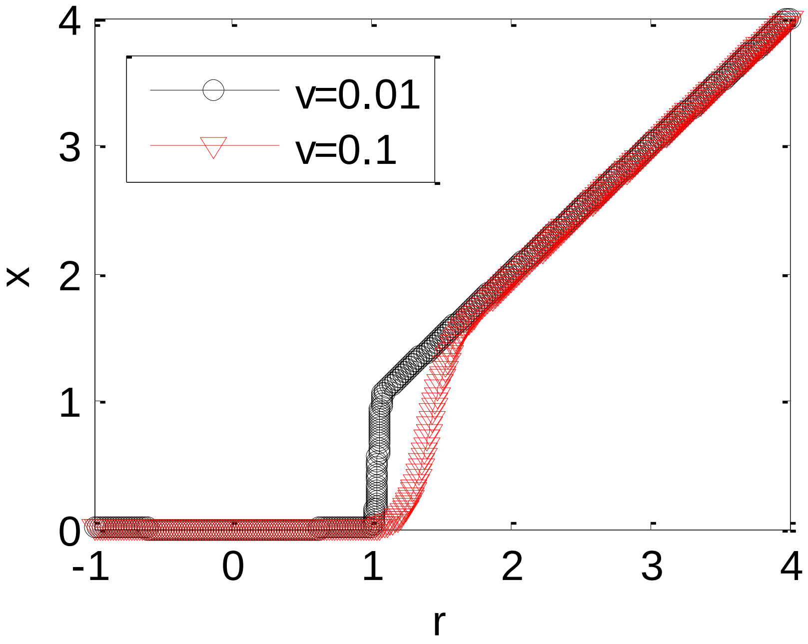
- Simulate the equation with *r* increasing linearly in time. Consider different variation rate (*v*) and/or different initial value of the parameter (r_0).

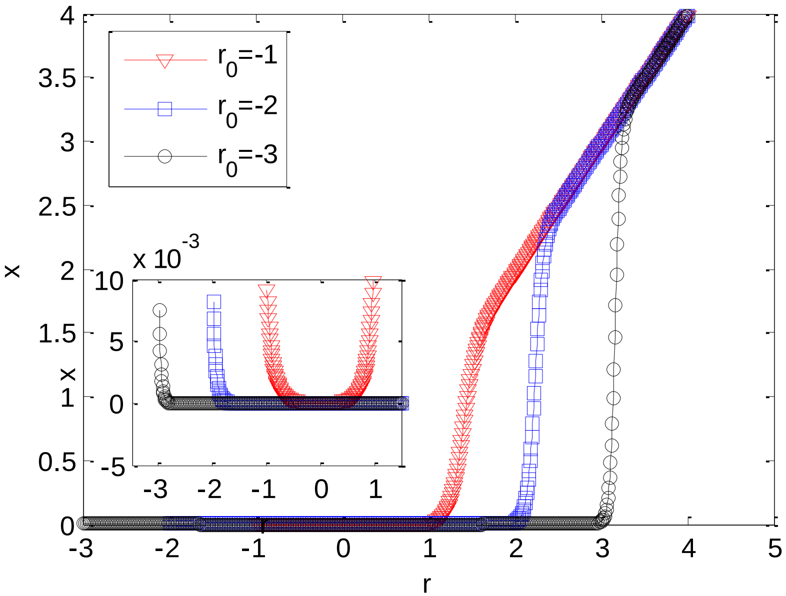
$$r(t) = r_0 + vt$$

$$x_0 = 0.01$$

$$v = 0.1$$

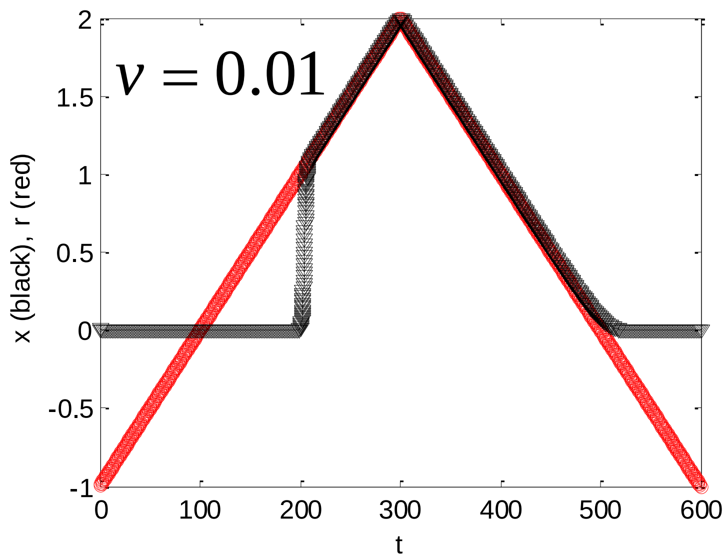
$$x_0 = 0.01$$

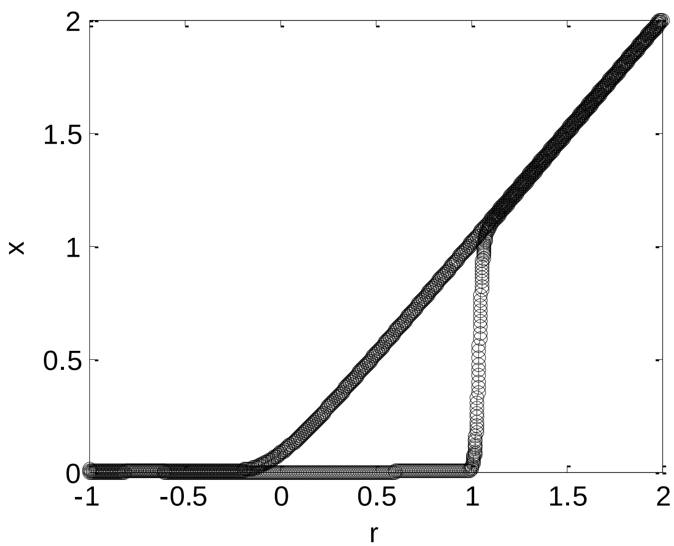




- Now consider that the control parameter *r* **increases and then decreases** linearly in time.

  Plot *x* and *r* vs time and plot *x* vs *r*.

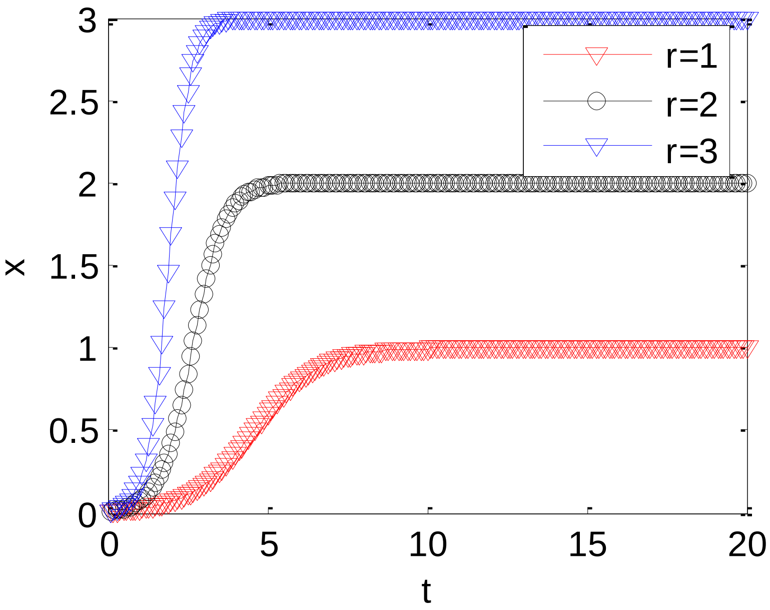




- Calculate the "turn on" when *r* is constant, $r>r^*=0$.

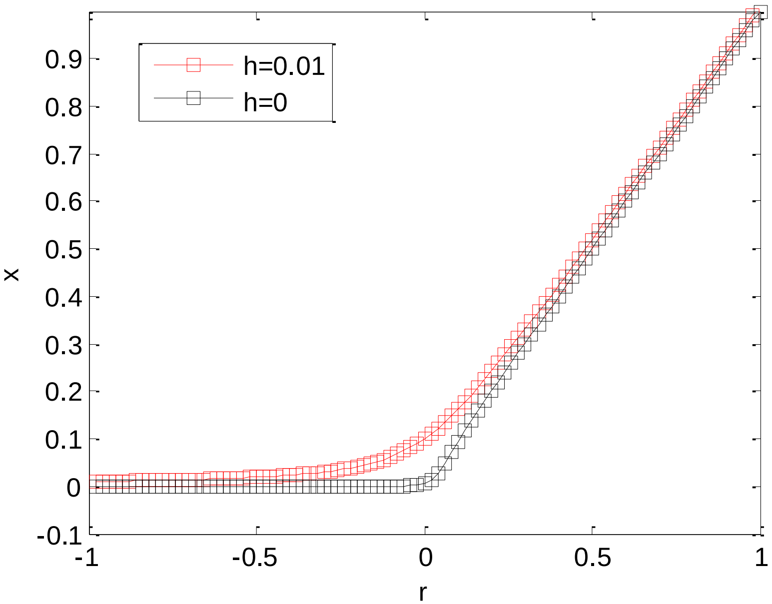
$$r(t) = r$$

$$x_0 = 0.01$$



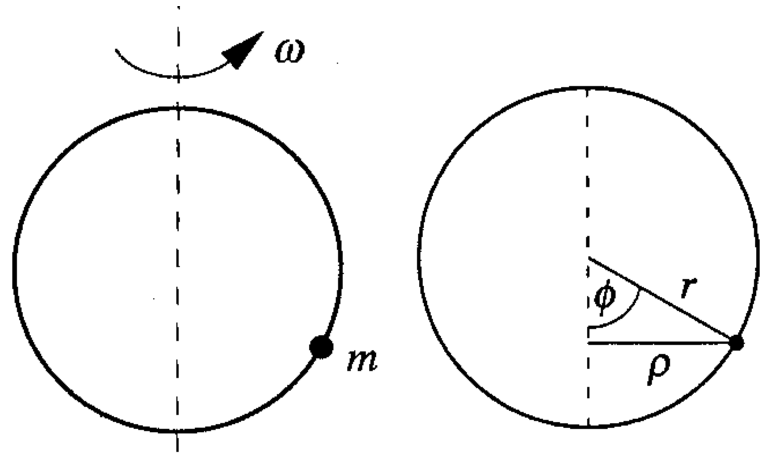
- Calculate the bifurcation diagram by plotting $x(t=50)$ vs *r*.

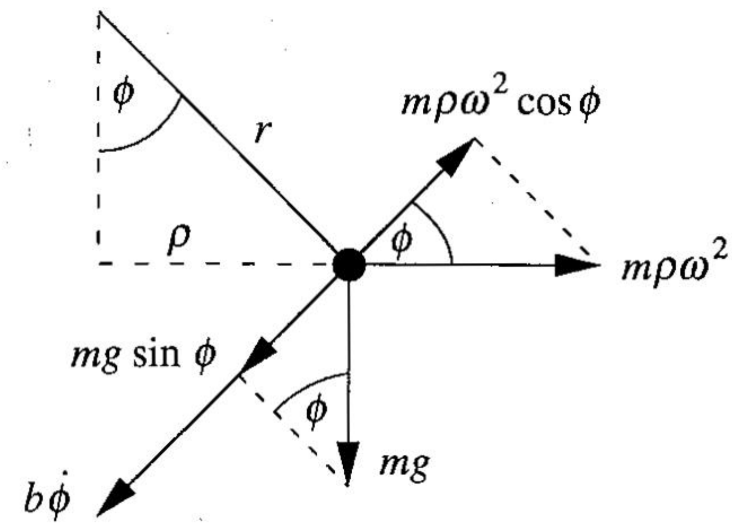
$$\dot{x} = rx - x^2 + h$$



Example: particle in a rotating wire hoop

- A particle moves along a wire hoop that rotates at constant angular velocity





$$mr\ddot{\phi} = -b\dot{\phi} - mg\sin\phi + mr\omega^2 \sin\phi\cos\phi$$

- Neglect the second derivative (more latter)

$$b\dot{\phi} = -mg\sin\phi + mr\omega^2 \sin\phi\cos\phi$$

$$= mg\sin\phi\left(\frac{r\omega^2}{g}\cos\phi - 1\right)$$

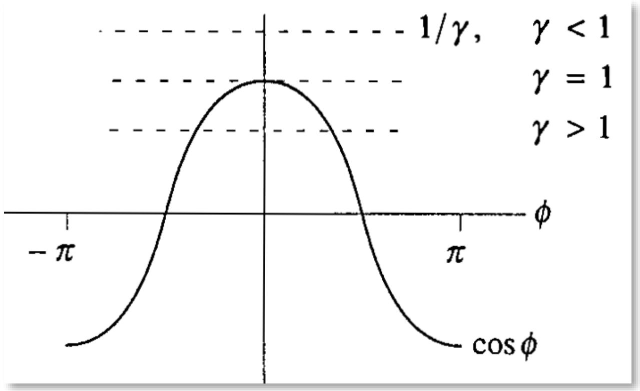
- Fixed points from: sin ϕ = 0

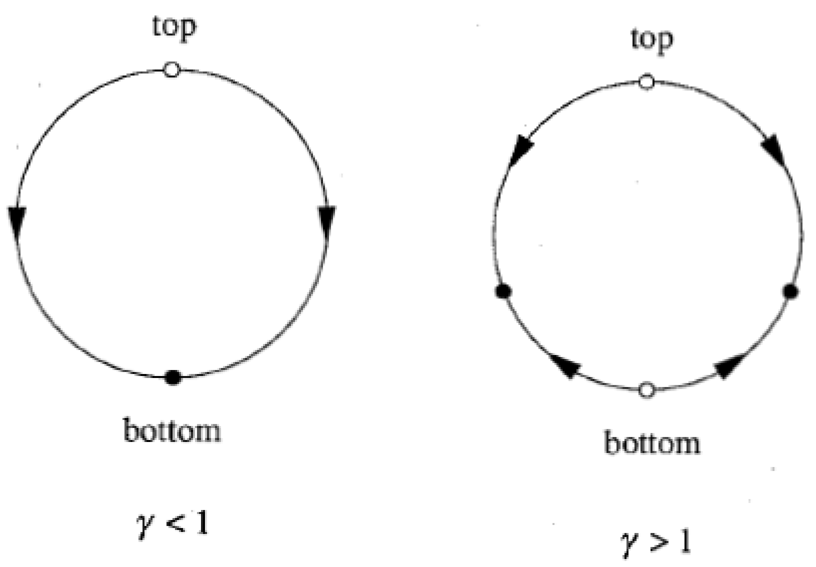
$\phi^* = 0$ (the bottom of the hoop) and $\phi^* = \pi$ (the top).

stable unstable

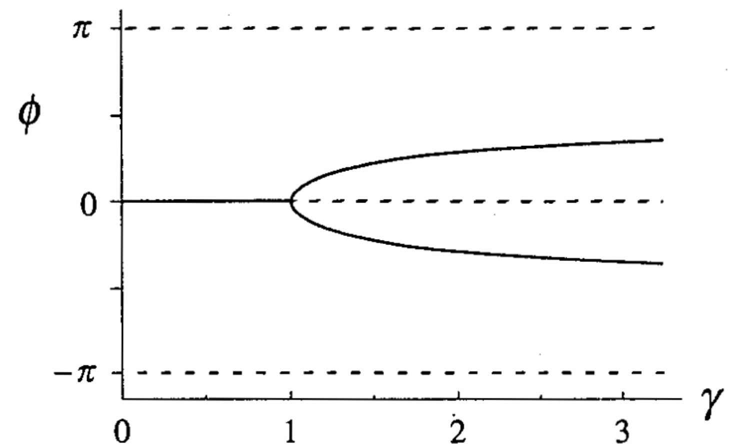
- Fixed points from: γ cos ϕ -1 = 0

$$\gamma = \frac{r\omega^2}{g}$$





Pitchfork Bifurcation:



When is this "first-order" description valid?

When is ok to neglect the second derivative d^2x/dt^2 ?

Dimensional analysis and scaling

Dimensionless equation

- Dimensionless time
  (T = characteristic time-scale) $\tau = \frac{t}{T}$

$$\left(\frac{r}{gT^2}\right)\frac{d^2\phi}{d\tau^2} = -\left(\frac{b}{mgT}\right)\frac{d\phi}{d\tau} - \sin\phi + \left(\frac{r\omega^2}{g}\right)\sin\phi\cos\phi$$

- We want the lhs very small, we define T such that

$$\frac{r}{gT^2} << 1 \quad \text{and} \quad \frac{b}{mgT} \approx O(1) \quad \Rightarrow \quad T = \frac{b}{mg}$$

$$\frac{r}{g}\left(\frac{mg}{b}\right)^2 << 1 \quad \Rightarrow \quad \boxed{b^2 >> m^2 gr}$$

- Define: $\varepsilon = \frac{m^2 gr}{b^2} \quad \Rightarrow \quad \boxed{\varepsilon\frac{d^2\phi}{d\tau^2} = -\frac{d\phi}{d\tau} - \sin\phi + \gamma\sin\phi\cos\phi} \qquad \gamma = \frac{r\omega^2}{g}$

Over damped limit

$$\varepsilon \frac{d^2\phi}{d\tau^2} = -\frac{d\phi}{d\tau} - \sin\phi + \gamma \sin\phi \cos\phi$$

- The dimension less equation suggests that the first-order equation is valid in the over damped limit: $\varepsilon \to 0$
- Problem: second-order equation has two independent initial conditions: $\phi(0)$ and $d\phi/d\tau(0)$
- But the first-order equation has only one initial condition $\phi(0)$, $d\phi/d\tau(0)$ is calculated from

$$\frac{d\phi}{d\tau} = -\sin\phi + \gamma \sin\phi \cos\phi$$

- Paradox: how can the first-order equation represent the second-order equation?

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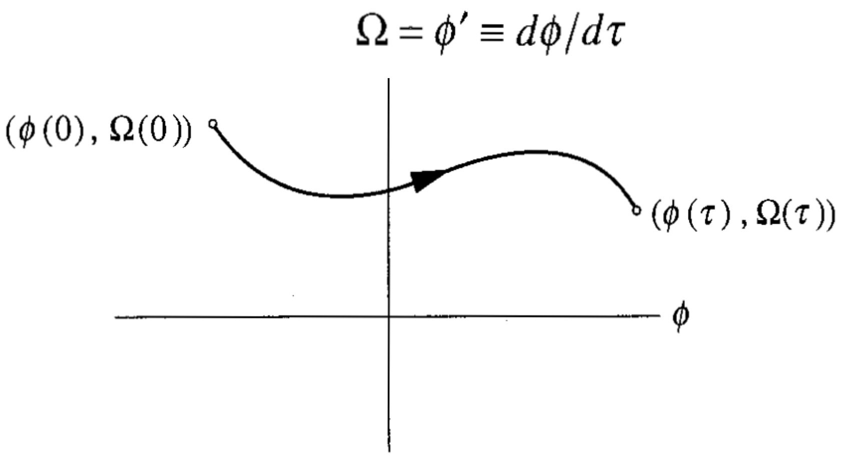
Trajectories in phase space

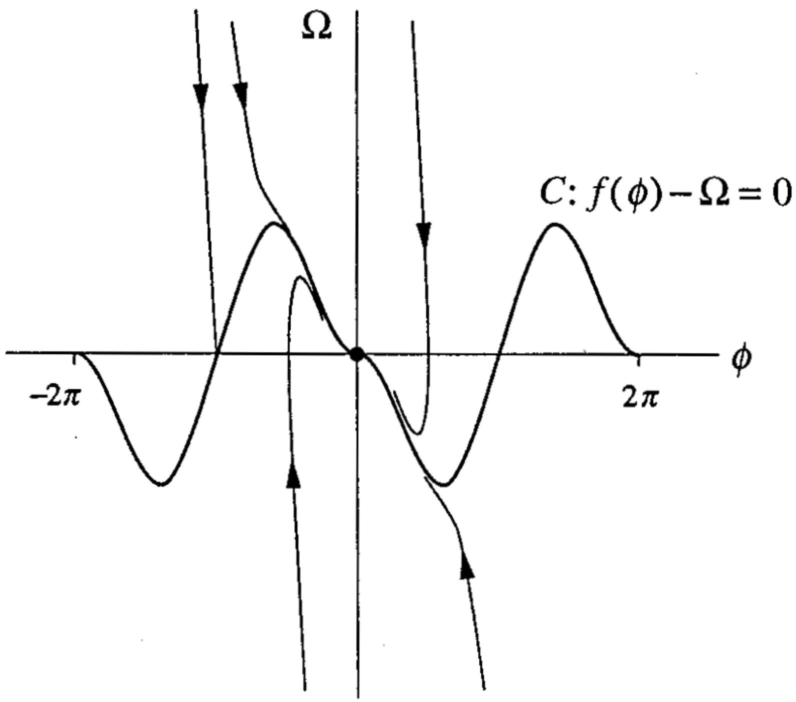
- First order system:

$$\frac{d\phi}{d\tau} = f(\phi) - \sin\phi + \gamma\sin\phi\cos\phi$$



- Second order system:

$$\varepsilon\frac{d^2\phi}{d\tau^2} = -\frac{d\phi}{d\tau} - \sin\phi + \gamma\sin\phi\cos\phi$$



Second order system:

$\varepsilon \to 0$

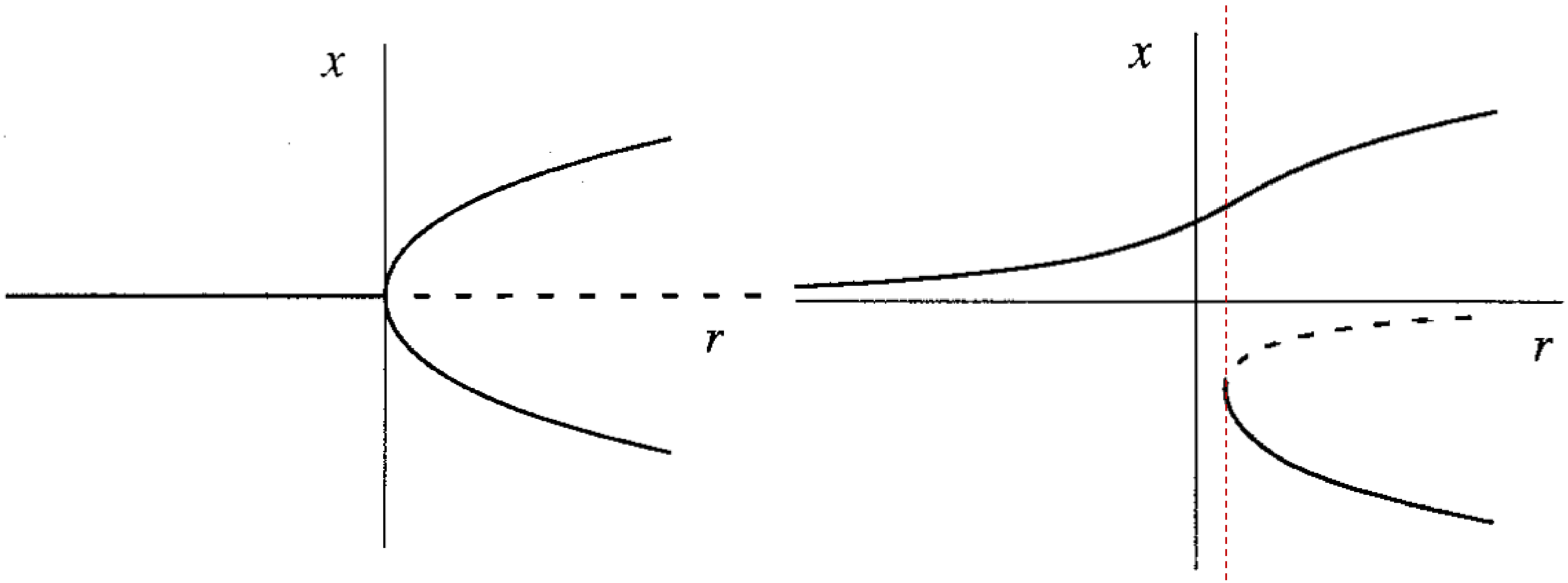
limit, *all trajectories slam straight up or down onto the curve C defined by* $f(\phi) = \Omega$, *and then slowly ooze along this curve until they reach a fixed point*

Outline

- Introduction to bifurcations
- Saddle-node, transcritical and pitchfork bifurcations
- Examples
- Imperfect bifurcations & catastrophes

Imperfect bifurcations

$$\dot{x} = h + rx - x^3$$

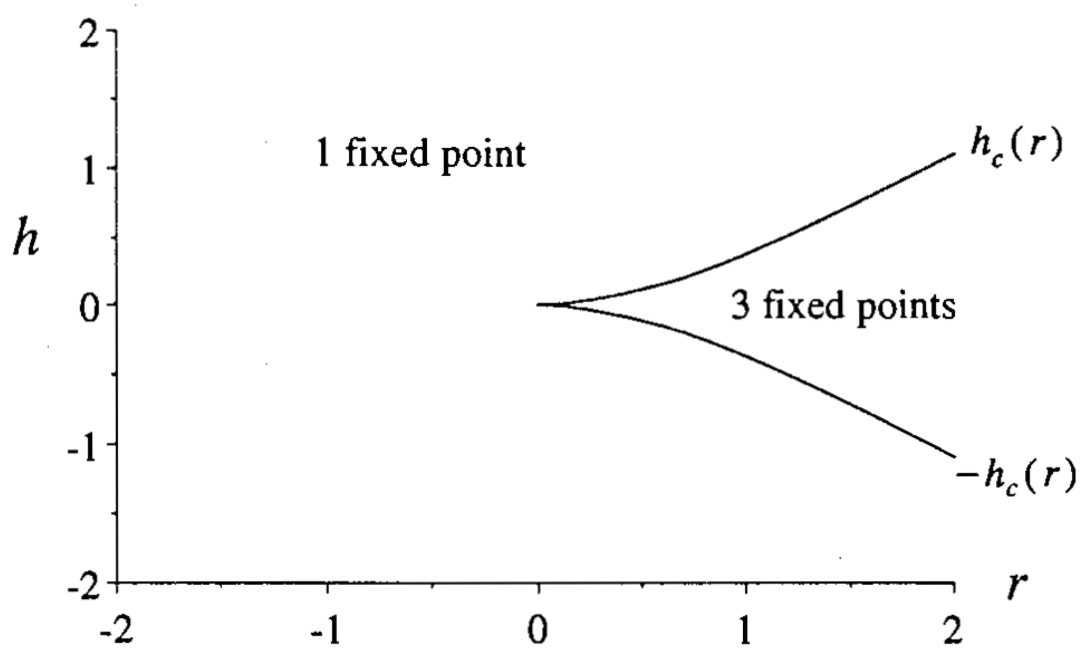


(a) $h = 0$

(b) $h \neq 0$

Parameter space (*h*, *r*)

$$\dot{x} = h + rx - x^3$$



Exercise : using these two equations

1. fixed points: $f(x^*) = 0$
2. saddle node bifurcation: $f'(x^*) = 0$

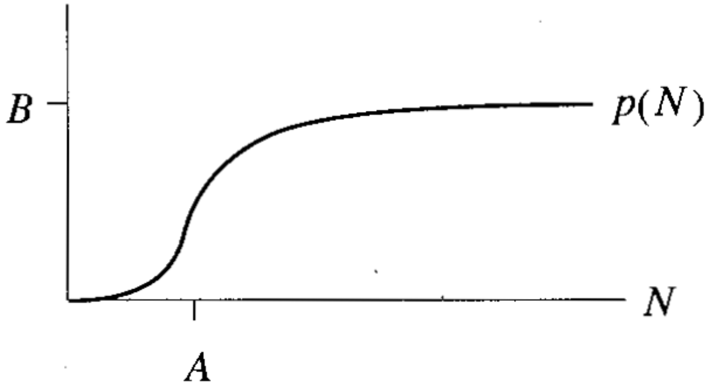
Calculate $h_c(r)$

$$h_c = \pm \frac{2r}{3}\sqrt{\frac{r}{3}}$$

Example: insect outbreak

$$\dot{N} = RN\left(1-\frac{N}{K}\right) - p(N)$$

- Budworms population grows logistically (*R*>0 grow rate)
- *p*(*N*): dead rate due to predation
- If no budworms (*N*≈0): no predation: birds look for food elsewhere
- If *N* large, *p*(*N*) saturates: birds eat as much as they can.



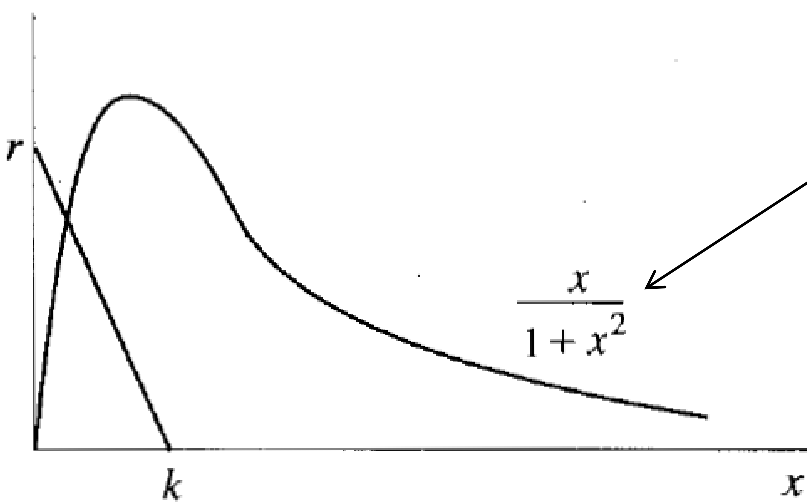
$$p(N) = \frac{BN^2}{A^2 + N^2} \qquad A, B > 0$$

Dimensionless formulation

$$x = N/A \qquad \tau = \frac{Bt}{A}, \qquad r = \frac{RA}{B}, \qquad k = \frac{K}{A}$$

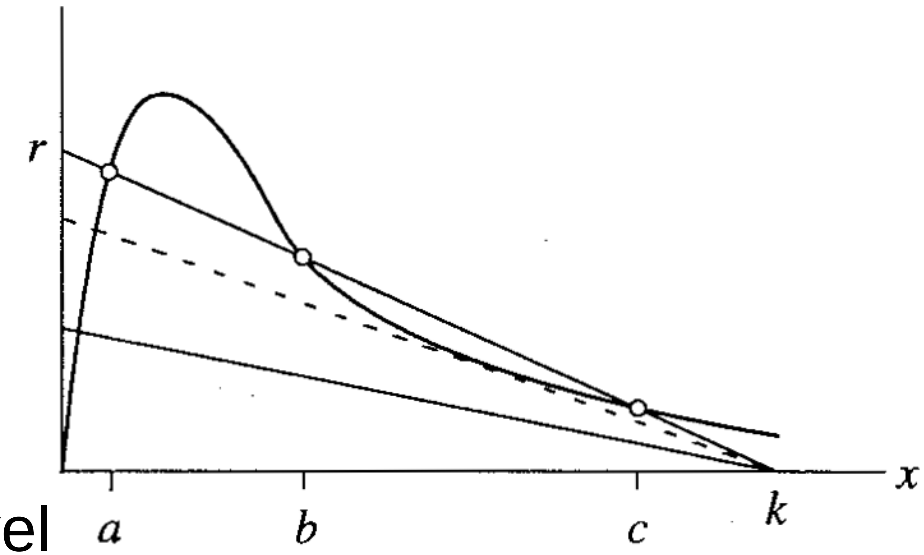
$$\frac{dx}{d\tau} = rx\left(1 - \frac{x}{k}\right) - \frac{x^2}{1+x^2}$$

- $x^*=0$
- Other FPs from the solution of $r\left(1 - \frac{x}{k}\right) = \frac{x}{1+x^2}$



Independent of r and k

- When the line intersects the curve tangentially (dashed line): saddle-node bifurcation



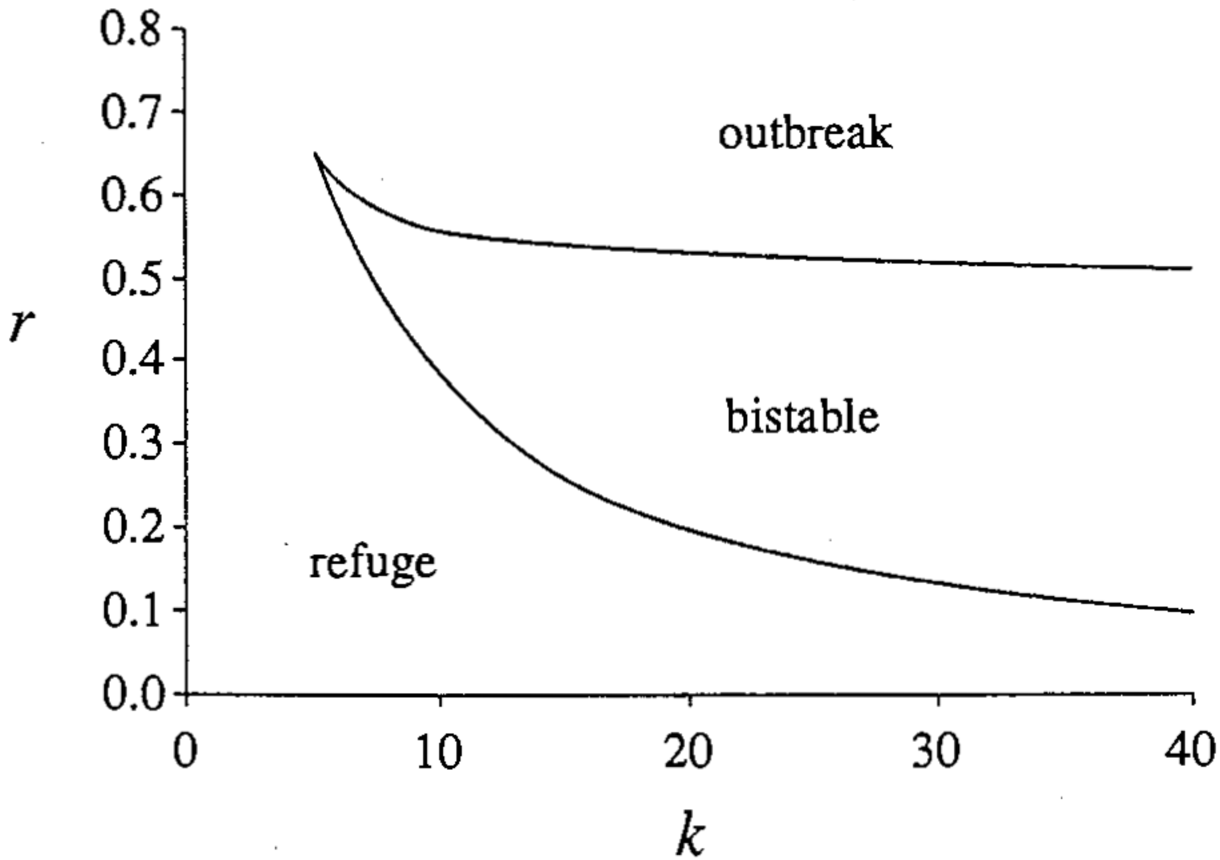
a: Refuge level of the budworm population

b: threshold

c: Outbreak level (pest)

Exercise : show that x*=0 is always unstable

Parameter space (*k*, *r*)



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