Nonlinear systems, chaos and control in Engineering

Bifurcations: saddle-node, transcritical and pitchfork

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Flows on the line = first-order ordinary differential equations.

\[ \frac{dx}{dt} = f(x) \]

Fixed point solutions: \( f(x^*) = 0 \)
- stable if \( f'(x^*) < 0 \)
- unstable if \( f'(x^*) > 0 \)
- neutral (bifurcation point) if \( f'(x^*) = 0 \)

There are no periodic solutions; the approach to fixed point solutions is monotonic (sigmoidal or exponential).
- Introduction to bifurcations
- Saddle-node, transcritical and pitchfork bifurcations
- Examples
- Imperfect bifurcations & catastrophes
A qualitative change (in the structure of the phase space) when a **control parameter is varied**:
- Fixed points can be created or destroyed
- The stability of a fixed point can change

There are many examples in physical systems, biological systems, etc.

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**What is a bifurcation?**
Example

Control parameter increases in time
Bifurcation and potential

Monostability

Bifurcation

Bistability
Bifurcations are not equivalent to qualitative change of behavior.

- Bifurcation but no change of behavior
- Change of behavior but no bifurcation
Introduction to bifurcations
Saddle-node, transcritical and pitchfork bifurcations
Examples
Imperfect bifurcations & catastrophes
Basic mechanism for the creation or the destruction of fixed points

\[ \dot{x} = f(x) = r + x^2 \]
\[ x^* = \pm \sqrt{-r} \]

At the bifurcation point \( r^* = 0 \):
\[ f'(x^*) = 0 \]
Calculate the fixed points and their stability as a function of the control parameter $r$

$x^* = \pm \sqrt{r}$

- $r < 0$
- $r = 0$
- $r > 0$
Normal forms

- Are representative of all saddle-node bifurcations.

- Close to the saddle-node bifurcation the dynamics can be approximated by

\[
\dot{x} = r - x^2 \quad \text{or} \quad \dot{x} = r + x^2
\]

Example:

\[
\dot{x} = r - x - e^{-x} = r - x - \left[1 - x + \frac{x^2}{2} + \cdots\right] = (r - 1) - \frac{x^2}{2} + \cdots
\]
Near a saddle-node bifurcation

$f(x)$ looks parabolic in here
A pair of fixed points appear (or disappear) out of the "clear blue sky" ("blue sky" bifurcation, Abraham and Shaw 1988).
Transcritical bifurcation

\[
\dot{x} = rx - x^2
\]

\[
x^* = 0
\]

\[
x^* = r
\]

are the fixed points for all \( r \)

(a) \( r < 0 \)

(b) \( r = 0 \)

(c) \( r > 0 \)

Transcritical bifurcation: general mechanism for changing the stability of fixed points.
\[ \dot{x} = rx - x^2 \]

fixed points \( x^* = 0 \) and \( x^* = r \)

\[ f'(x) = r - 2x \]

\[ f'(0) = r \]

\[ f'(r) = -r \]

- Exchange of stability at \( r = 0 \).

- **Exercise:** \( \dot{x} = r \ln x + x - 1 \)
  show that a transcritical bifurcation occurs near \( x=1 \)
  (hint: consider \( u = x-1 \) small)
Pitchfork bifurcation

\[ \dot{x} = rx - x^3 \]

Symmetry \( x \rightarrow -x \)

One fixed point \( \rightarrow \) 3 fixed points
The governing equation is symmetric: $x \rightarrow -x$ but for $r > 0$: symmetry broken solutions.
\[ \dot{x} = rx - x^3 \quad \dot{x} = f(x) \quad f(x) = -\frac{dV}{dx} \quad V(x) = -\frac{1}{2} rx^2 + \frac{1}{4} x^4 \]
Pitchfork bifurcations

Supercritical: \( x^3 \) is stabilizing

\[ \dot{x} = rx - x^3 \]

Subcritical: \( x^3 \) is destabilizing

\[ \dot{x} = rx + x^3 \]
Exercise: find the fixed points and compute their stability

\[ \dot{x} = rx + x^3 - x^5 \]
Subcritical bifurcation: Hysteresis

Critical or dangerous transition! A lot of effort in trying to find "early warning signals" (more latter)
Hysteresis: sudden changes in visual perception

Fischer (1967): experiment with 57 students.

“When do you notice an abrupt change in perception?”
Bifurcation condition: change in the stability of a fixed point

\[ f'(x^*) = 0 \]

In first-order ODEs: three possible bifurcations
- Saddle node
- Pitchfork
- Trans-critical

The normal form describes the behavior near the bifurcation.
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Example: neuron model

\[ C \dot{V} = I - g_L(V - E_L) - g_{Na} m_\infty(V)(V - E_{Na}) \]

\[ m_\infty(V) = \frac{1}{1 + \exp\left\{ \frac{(V_{1/2} - V)}{k} \right\}} \]

\[ C = 10 \, \mu F, \quad I = 0 \, \text{pA}, \quad g_L = 19 \, \text{mS}, \quad E_L = -67 \, \text{mV}, \]
\[ g_{Na} = 74 \, \text{mS}, \quad V_{1/2} = 1.5 \, \text{mV}, \quad k = 16 \, \text{mV}, \quad E_{Na} = 60 \, \text{mV} \]
Saddle-node Bifurcation

- **I=16**
  - Bifurcation diagram for $I=16$
  - Excited state shown

- **I=60**
  - Bifurcation diagram for $I=60$
  - Excited state shown

**Graphs:**
- $F(V)$ vs. membrane potential, $V$ (mV)
- Membrane potential, $V(t)$ vs. time (ms)
Near the bifurcation point: slow dynamics

This slow transition is an “early warning signal” of a critical or dangerous transition ahead (more latter)
If the control parameter now decreases
Simulate the neuron model with different values of the control parameter $I$ and/or different initial conditions.
\[ \dot{n} = \text{gain} - \text{loss} \]
\[ = GnN - kn. \]
\[ N(t) = N_0 - \alpha n \]
\[ \dot{n} = Gn(N_0 - \alpha n) - kn \]
\[ = (GN_0 - k)n - (\alpha G)n^2 \]
\[ \dot{x} = rx - x^2 \]
Transcritical Bifurcation

\[ n \]

\[ k/G \]

\[ N_0 \]

lamp

laser
“imperfect” bifurcation due to noise

Fig. 1.17 Imperfect bifurcation for a laser in the presence of spontaneous emission, measured for a He-Ne laser. Reprinted Figure 1 with permission from Corti and Degiorgio [42]. Copyright 1976 by the American Physical Society.
Fig. 1.3 He-Ne gas laser output as a function of time. From the lower to the upper time traces, the pump parameter above threshold is gradually increased. Reprinted Figure 2 with permission from Pariser and Marshall [30]. Copyright 1965 by the American Institute of Physics.
Laser turn-on delay

\[ \dot{x} = rx - x^2 \]

\[ r(t) = r_0 + vt \]

\[ r_0 < r^* = 0 \]

Linear increase of control parameter

Start before the bifurcation point
Comparison with experimental observations

**Dynamical hysteresis**

Quasi-static very slow variation of the control parameter

Am. J. Phys., Vol. 72, No. 6, June 2004
Simulate the equation with $r$ increasing linearly in time. Consider different variation rate ($v$) and/or different initial value of the parameter ($r_0$).

$$\dot{x} = r x - x^2$$

- $r(t) = r_0 + vt$
- $x_0 = 0.01$
- $v = 0.1$
- $x_0 = 0.01$
Now consider that the control parameter $r$ increases and then decreases linearly in time. Plot $x$ and $r$ vs time and plot $x$ vs $r$. 

![Graph showing $v = 0.01$]
- Calculate the “turn on” when $r$ is constant, $r > r^* = 0$.

\[ r(t) = r \]

\[ x_0 = 0.01 \]

- Calculate the bifurcation diagram by plotting $x(t=50)$ vs $r$.

\[ \dot{x} = rx - x^2 + h \]
A particle moves along a wire hoop that rotates at constant angular velocity.

\[
m r \ddot{\phi} = -b \dot{\phi} - mg \sin \phi + m r \omega^2 \sin \phi \cos \phi
\]
Neglect the second derivative (more latter)

\[ \dot{b}\phi = -mg \sin \phi + mr\omega^2 \sin \phi \cos \phi \]

\[ = mg \sin \phi \left( \frac{r\omega^2}{g} \cos \phi - 1 \right) \]

- Fixed points from: \( \sin \phi = 0 \)
  \[ \phi^* = 0 \text{ (the bottom of the hoop)} \] and \( \phi^* = \pi \text{ (the top)} \)
  stable

- Fixed points from: \( \gamma \cos \phi - 1 = 0 \)
  \[ \gamma = \frac{r\omega^2}{g} \]
When is this “first-order” description valid?
When is ok to neglect the second derivative \( \frac{d^2x}{dt^2} \)?

Dimensional analysis and scaling
Dimensionless time

\( (T = \text{characteristic time-scale}) \)

\[ \tau = \frac{t}{T} \]

\[ \left( \frac{r}{gT^2} \right) \frac{d^2 \phi}{d\tau^2} = -\left( \frac{b}{mgT} \right) \frac{d\phi}{d\tau} - \sin \phi + \left( \frac{r\omega^2}{g} \right) \sin \phi \cos \phi \]

- We want the lhs very small, we define \( T \) such that

\[ \frac{r}{gT^2} \ll 1 \quad \text{and} \quad \frac{b}{mgT} \approx O(1) \quad \Rightarrow \quad T = \frac{b}{mg} \]

\[ \frac{r}{g} \left( \frac{mg}{b} \right)^2 \ll 1 \quad \Rightarrow \quad b^2 >> m^2 gr \]

- Define: \( \varepsilon = \frac{m^2 gr}{b^2} \)

\[ \varepsilon \frac{d^2 \phi}{d\tau^2} = -\frac{d\phi}{d\tau} - \sin \phi + \gamma \sin \phi \cos \phi \]

\[ \gamma = \frac{r\omega^2}{g} \]
Over damped limit

\[ \varepsilon \frac{d^2 \phi}{d \tau^2} = - \frac{d \phi}{d \tau} - \sin \phi + \gamma \sin \phi \cos \phi \]

- The dimension less equation suggests that the first-order equation is valid in the over damped limit: \( \varepsilon \to 0 \)
- Problem: second-order equation has two independent initial conditions: \( \phi(0) \) and \( d\phi/d\tau(0) \)
- But the first-order equation has only one initial condition \( \phi(0), \frac{d\phi}{d\tau}(0) \) is calculated from
  \[ \frac{d\phi}{d\tau} = -\sin \phi + \gamma \sin \phi \cos \phi \]
- Paradox: how can the first-order equation represent the second-order equation?
First order system:

\[
\frac{d\phi}{d\tau} = f(\phi) - \sin \phi + \gamma \sin \phi \cos \phi
\]

\[
\Omega = \phi' \equiv \frac{d\phi}{d\tau}
\]

\[
(f(0), \Omega(0)) \quad \quad \quad (\phi(\tau), \Omega(\tau))
\]

Second order system:

\[
\varepsilon \frac{d^2\phi}{d\tau^2} = -\frac{d\phi}{d\tau} - \sin \phi + \gamma \sin \phi \cos \phi
\]

\[
C: f(\phi) - \Omega = 0
\]

Second order system:

\[\varepsilon \to 0\]

limit, all trajectories slam straight up or down onto the curve C defined by \(f(\phi) = \Omega\), and then slowly ooze along this curve until they reach a fixed point.
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Imperfect bifurcations

\[ \dot{x} = h + rx - x^3 \]

(a) \( h = 0 \)

(b) \( h \neq 0 \)
Exercise: using these two equations
1. fixed points: \( f(x^*) = 0 \)
2. saddle node bifurcation: \( f'(x^*) = 0 \)
Calculate \( h_c(r) \)

\[ h_c = \pm \frac{2r}{3} \sqrt{\frac{r}{3}} \]
Example: insect outbreak

\[ \dot{N} = RN \left( 1 - \frac{N}{K} \right) - p(N) \]

- Budworms population grows logistically \((R > 0 \text{ grow rate})\)
- \(p(N)\): dead rate due to predation
- If no budworms \((N \approx 0)\): no predation: birds look for food elsewhere
- If \(N\) large, \(p(N)\) saturates: birds eat as much as they can.

\[ p(N) = \frac{BN^2}{A^2 + N^2} \quad A, B > 0 \]
Dimensionless formulation

\[ x = \frac{N}{A} \quad \tau = \frac{Bt}{A}, \quad r = \frac{RA}{B}, \quad k = \frac{K}{A} \]

\[ \frac{dx}{d\tau} = rx \left(1 - \frac{x}{k}\right) - \frac{x^2}{1 + x^2} \]

- \( x^* = 0 \)
- Other FPs from the solution of \( r \left(1 - \frac{x}{k}\right) = \frac{x}{1 + x^2} \)

Independent of \( r \) and \( k \)
When the line intersects the curve tangentially (dashed line): saddle-node bifurcation

Exercise: show that \( x^* = 0 \) is always unstable
Parameter space \((k, r)\)

• Thomas Erneux and Pierre Glorieux: *Laser Dynamics* (Cambridge University Press 2010)