Nonlinear systems, chaos and control in Engineering

Phase oscillators, entrainment and locking

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Outline

- **Introduction to phase oscillators**
- Nonlinear oscillator: the Adler equation
- Entrainment and locking

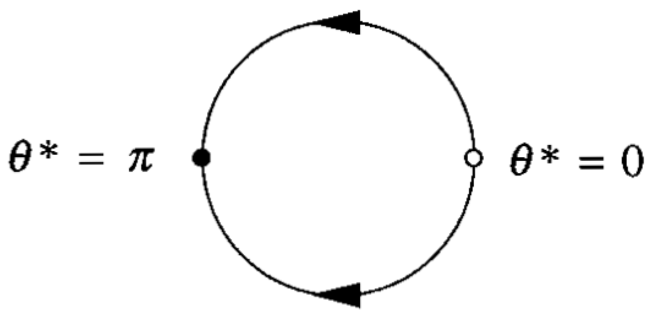
Phase oscillator

$$\dot{\theta} = f(\theta)$$

- Flow on a circle: a function that assigns a unique velocity vector to each point on the circle.

$$f(\theta + 2\pi) = f(\theta)$$

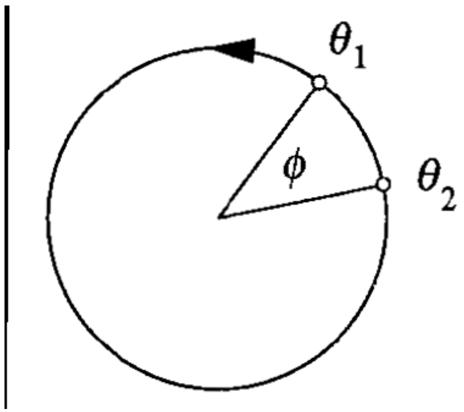
- Example: $\dot{\theta} = \sin\theta$



Linear oscillator: $\dot{\theta} = \omega$

EXAMPLE 4.2.1:

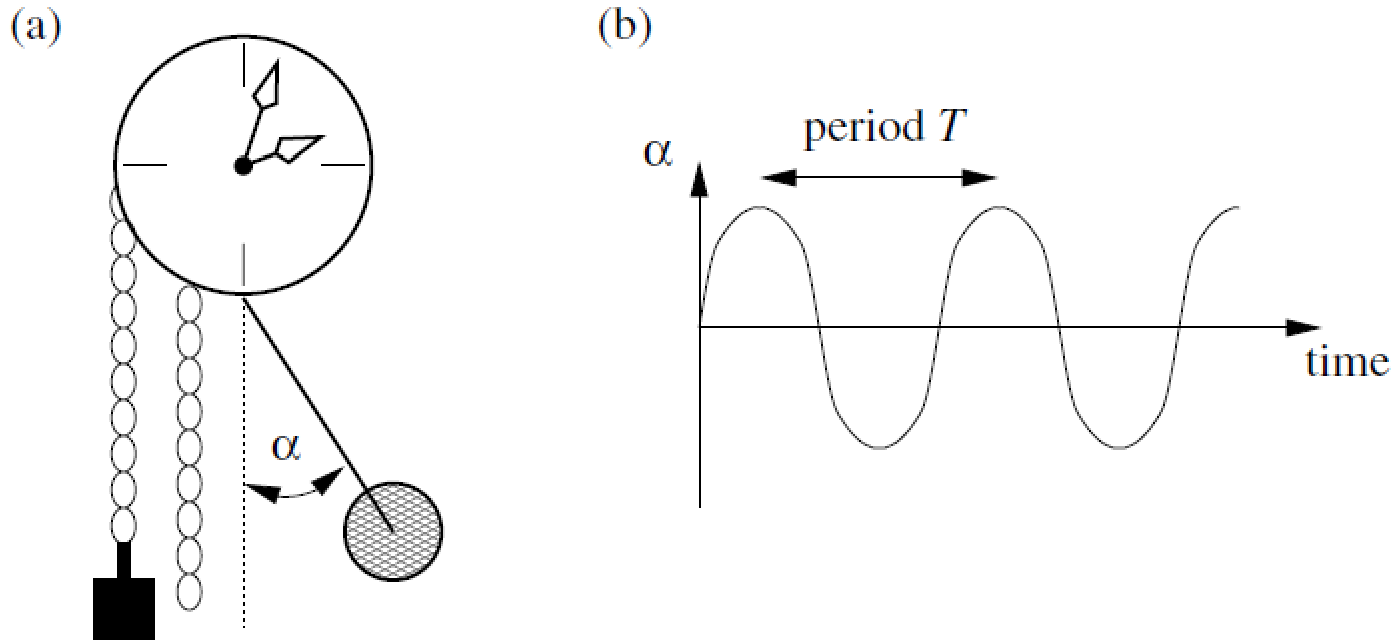
Two joggers, Speedy and Pokey, are running at a steady pace around a circular track. It takes Speedy T_1 seconds to run once around the track, whereas it takes Pokey $T_2 > T_1$ seconds. Of course, Speedy will periodically overtake Pokey; how long does it take for Speedy to lap Pokey once, assuming that they start together?



$$T_{\text{lap}} = \frac{2\pi}{\omega_1 - \omega_2} = \left(\frac{1}{T_1} - \frac{1}{T_2}\right)^{-1}$$

- Two **non-interacting** oscillators periodically go in and out of phase
- Beat frequency = $1/T_{lap}$

Example of a phase oscillator



Interacting oscillators

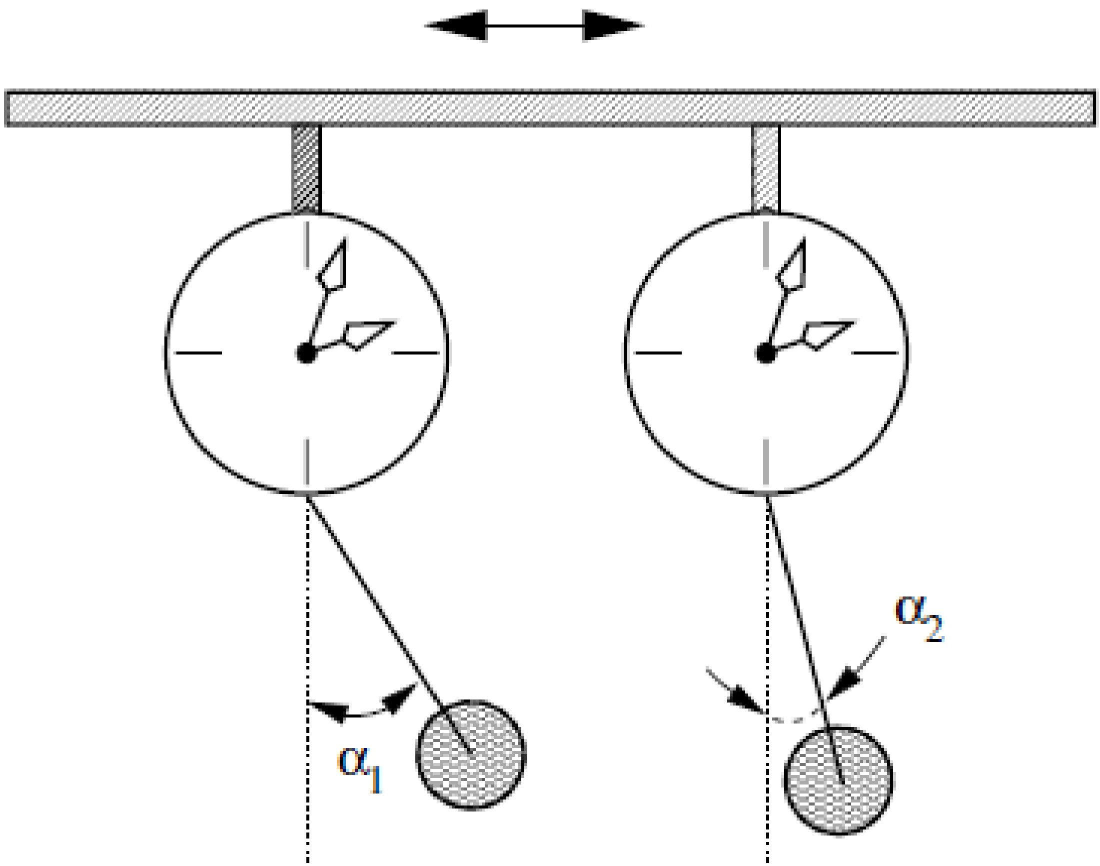


Figure 1.8. Two pendulum clocks coupled through a common support. The beam to which the clocks are fixed is not rigid, but can vibrate slightly, as indicated by the arrows at the top of the figure. This vibration is caused by the motions of both pendula; as a result the two clocks "feel" the presence of each other.

In phase vs out of phase oscillation

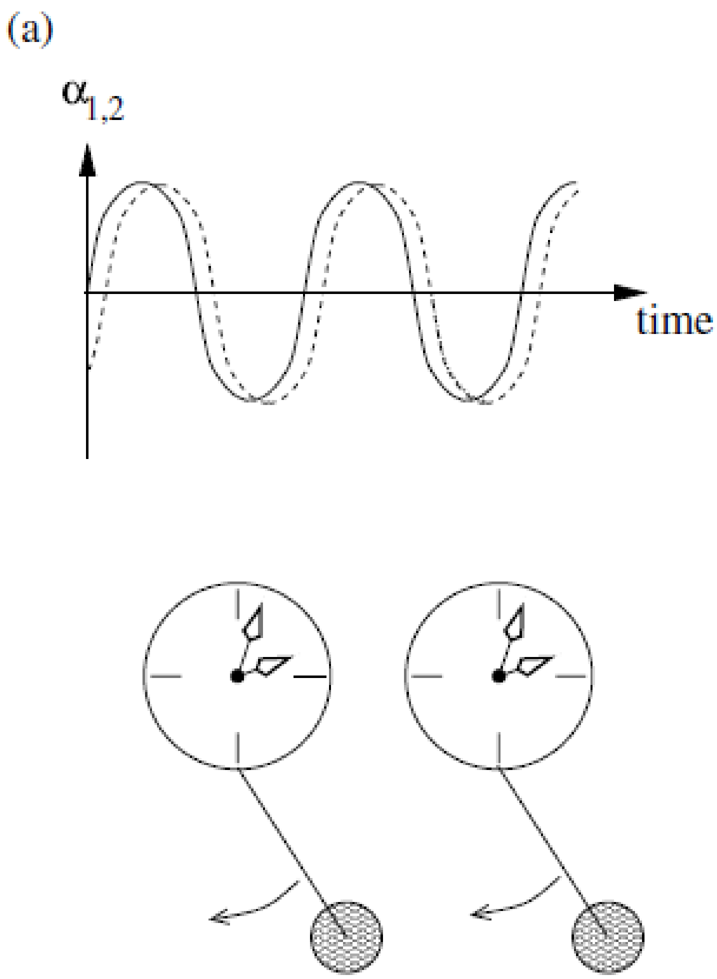
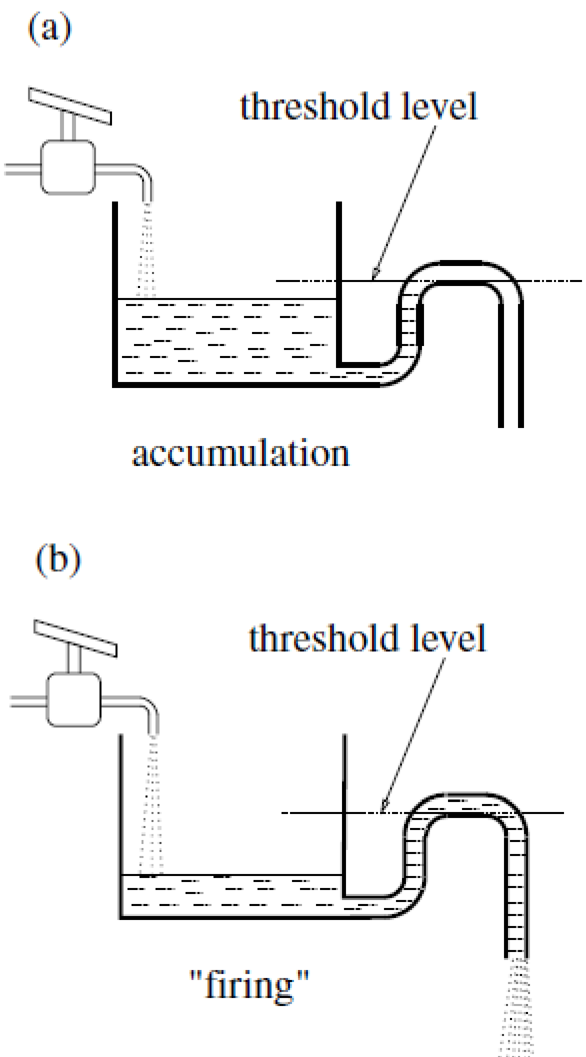


Figure 1.10. Possible synchronous regimes of two nearly identical clocks: they may be synchronized almost in-phase (a), i.e., with the phase difference $\phi_2 - \phi_1 \approx 0$, or in anti-phase (b), when $\phi_2 - \phi_1 \approx \pi$.

Example: Integrate (accumulate) and fire oscillator



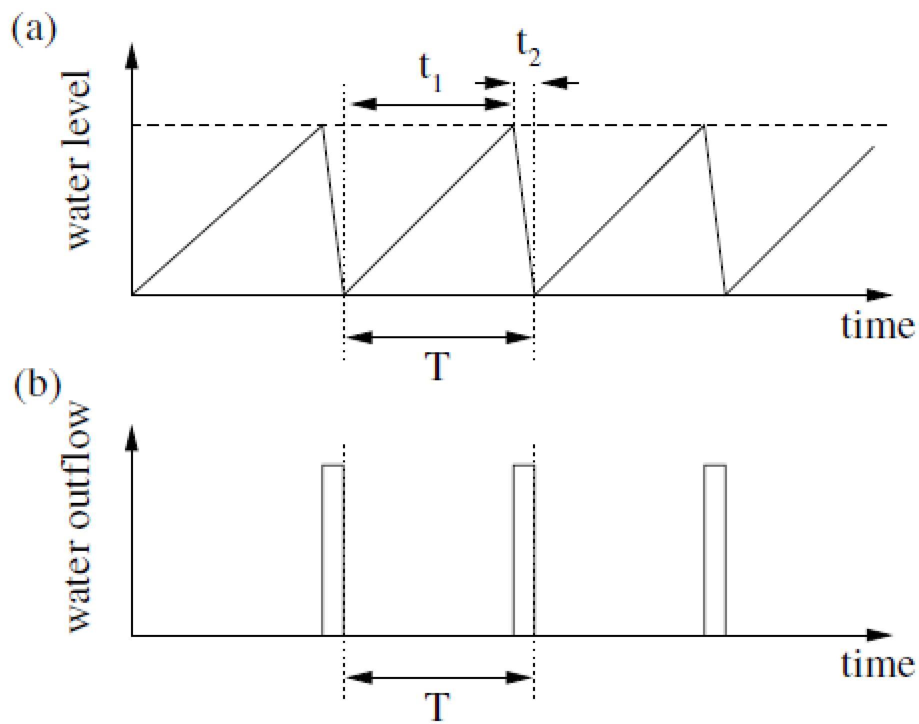


Figure 2.13. Time course of the mechanical integrate-and-fire (accumulate-and-fire) oscillator shown in Fig. 2.12. Water accumulates until it reaches the threshold level shown by a dashed line (a), then the water level is quickly reset to zero. The resetting corresponds to the pulse in the plot of the water outflow from the trap (b).

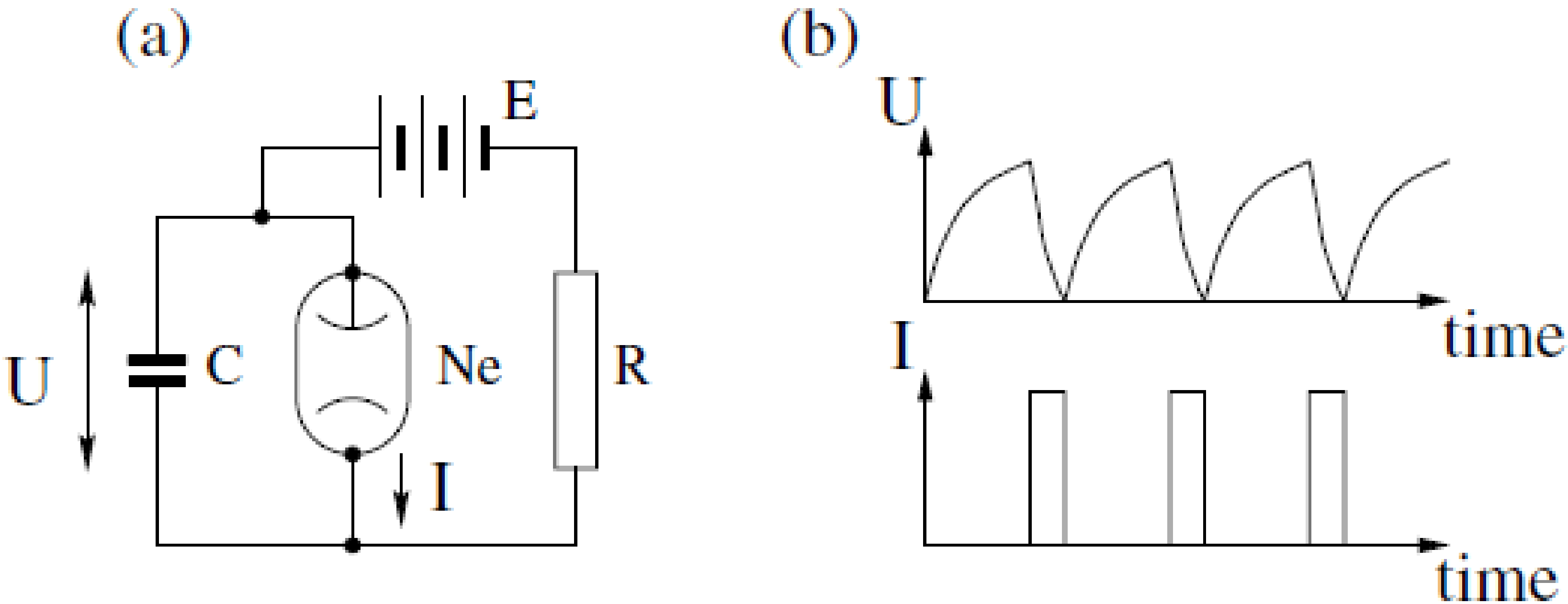


Figure 2.14. Schematic diagram of the van der Pol relaxation generator (a). The voltage at the capacitor increases until it reaches a threshold value at which the tube becomes conductive; then, the capacitor quickly discharges, the tube flashes, and a short pulse of current through the tube is observed. Each oscillatory cycle thus consists of epochs of accumulation and firing (b).

- Other examples: heart beats, neuronal spikes

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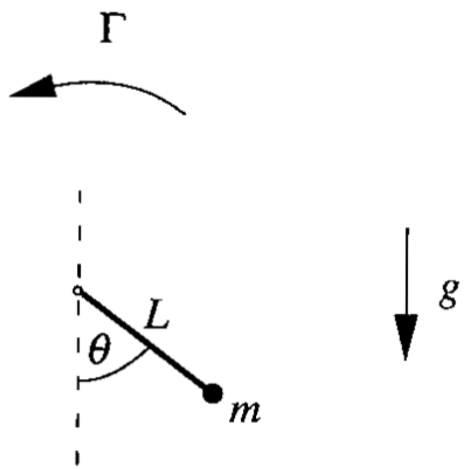
- **Nonlinear oscillator: the Adler equation**
- Entrainment and locking

An overdamped pendulum driven by a constant torque

$$mL^2\ddot{\theta} + b\dot{\theta} + mgL\sin\theta = \Gamma$$



- *b* very large:

$$b\dot{\theta} + mgL\sin\theta = \Gamma$$

- Dimension-less equation:

$$\frac{b}{mgL}\dot{\theta} = \frac{\Gamma}{mgL} - \sin\theta$$

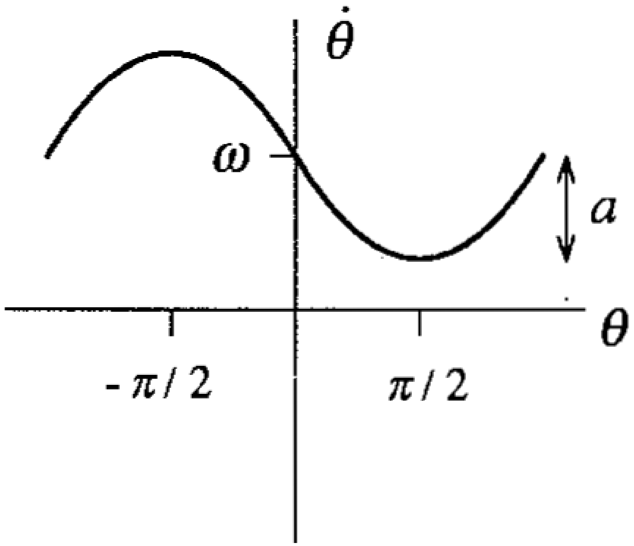
$$\tau = \frac{mgL}{b}t, \qquad \gamma = \frac{\Gamma}{mgL}$$

$$\theta' = \gamma - \sin\theta$$

$$\theta' = d\theta/d\tau$$

Nonlinear oscillator

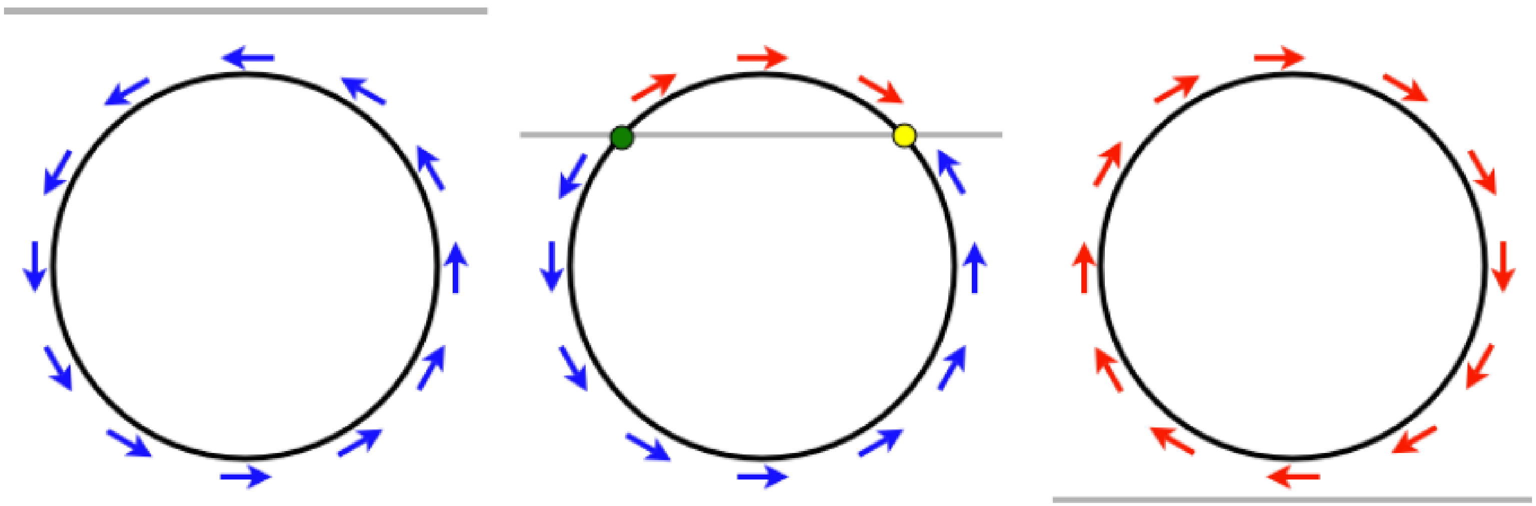
$$\dot{\theta} = \omega - a \sin \theta$$



- (Adler equation) Simple model for many nonlinear phase oscillators (neurons, circadian rhythms, over-damped driven pendulum), etc.

Robert Adler (1913–2007) is best known as the co-inventor of the **television remote control** using ultrasonic waves. But in the 1940s, he and others at Zenith Corporation were interested in reducing the number of vacuum tubes in an FM radio. The possibility that a locked oscillator might offer a solution inspired his 1946 paper "A Study of Locking Phenomena in Oscillators."

$$\dot{\theta} = \omega - \sin\theta$$

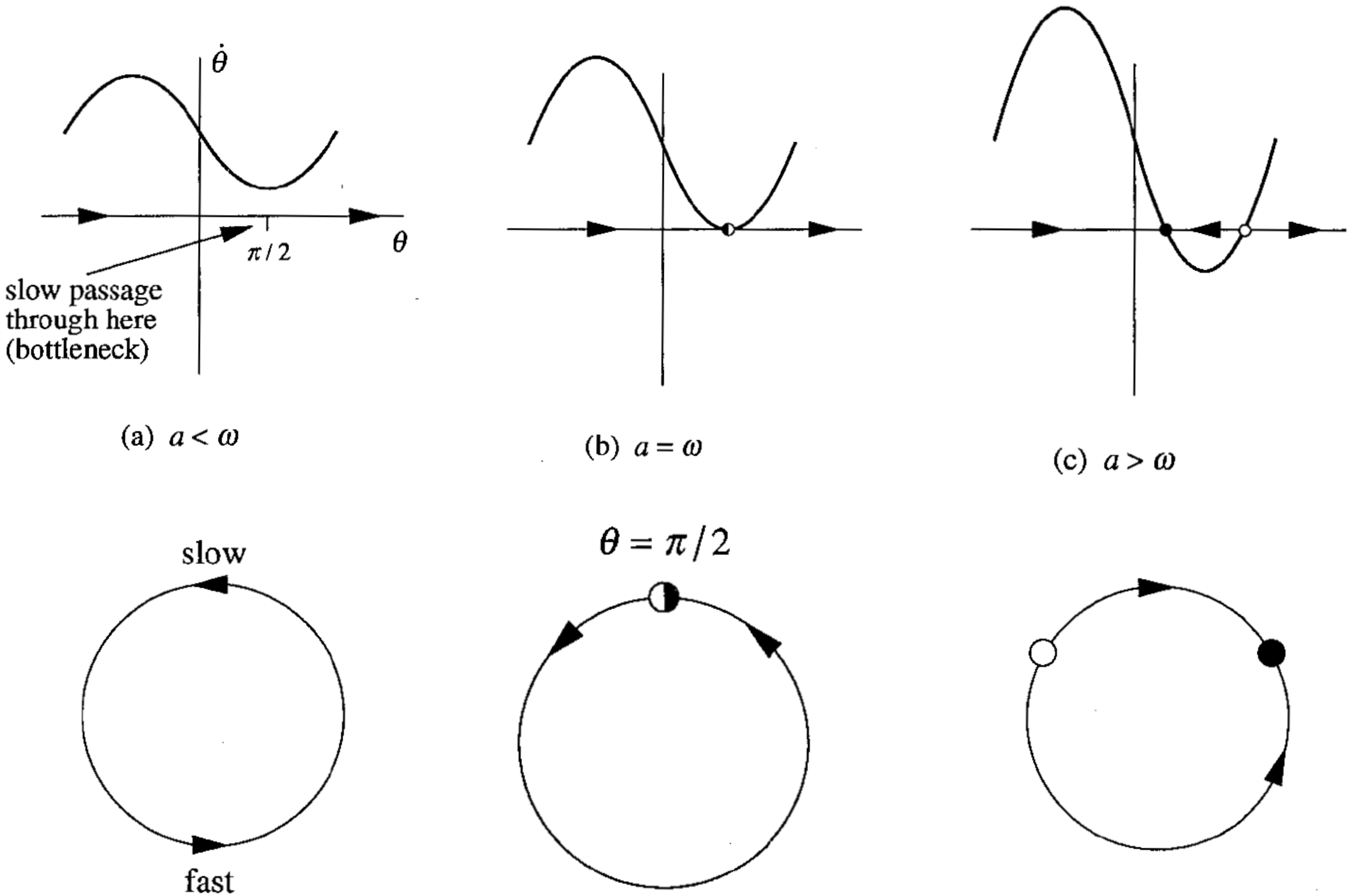


$\omega = 1.2$ $\quad\quad$ $\omega = \frac{1}{\sqrt{2}}$ $\quad\quad$ $\omega = -1.2$

For $|\omega|<1$: simple model of an **excitable** system:

- With a small perturbation: fast return to the stable state
- But if the perturbation in larger than a threshold, then, long "excursion" before returning to the stable state.

$$\dot{\theta} = \omega - a\sin\theta$$



Fixed points when *a* > ω

$$\dot{\theta} = \omega - a\sin\theta$$

$$\sin\theta^* = \omega/a \qquad \cos\theta^* = \pm\sqrt{1-(\omega/a)^2}$$

- Linear stability:

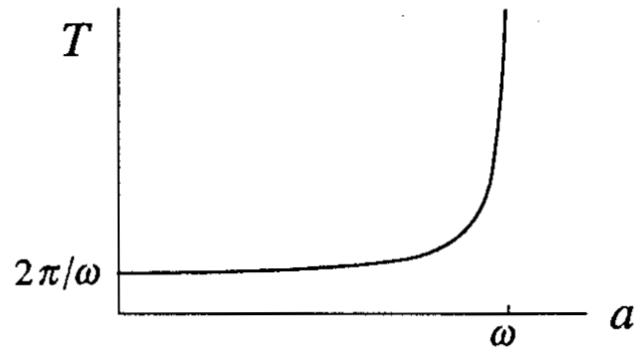
$$f'(\theta^*) = -a\cos\theta^* = \mp a\sqrt{1-(\omega/a)^2}$$

- The FP with cos θ*>0 is the stable one.

Oscillation period when *a* < ω

$$\dot{\theta} = \omega - a\sin\theta$$

$$T = \int dt = \int_0^{2\pi} \frac{dt}{d\theta}\, d\theta = \int_0^{2\pi} \frac{d\theta}{\omega - a\sin\theta} \qquad T = \frac{2\pi}{\sqrt{\omega^2 - a^2}}$$



a=0: uniform oscillator

T→∞ when *a*→ω:

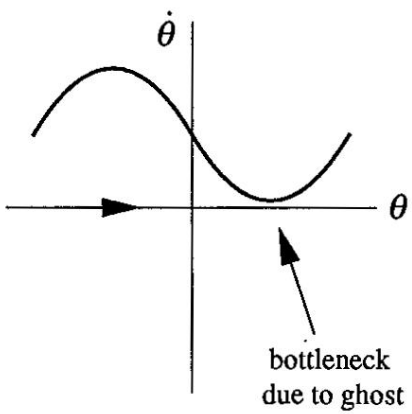
Bottleneck

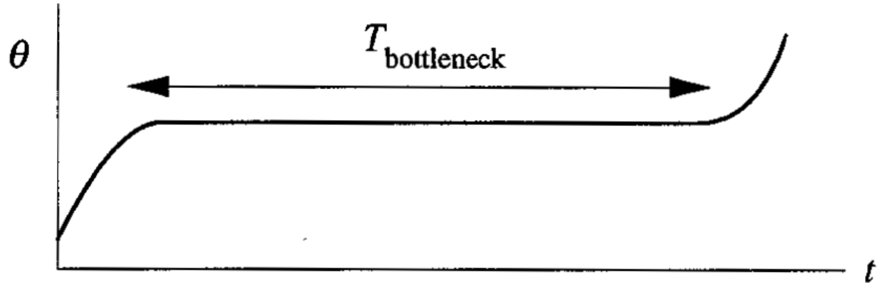
$$a \to \omega^{-} \qquad T \approx \left(\frac{\pi\sqrt{2}}{\sqrt{\omega}}\right)\frac{1}{\sqrt{\omega - a}} \qquad a_c = \omega$$

- Period grows to infinite as $(a_c - a)^{-1/2}$

  "**critical slowing down**": early warning signal of a critical transition ahead.

- Generic feature at a saddle-node bifurcation





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Synchronous rhythmic flashing of fireflies

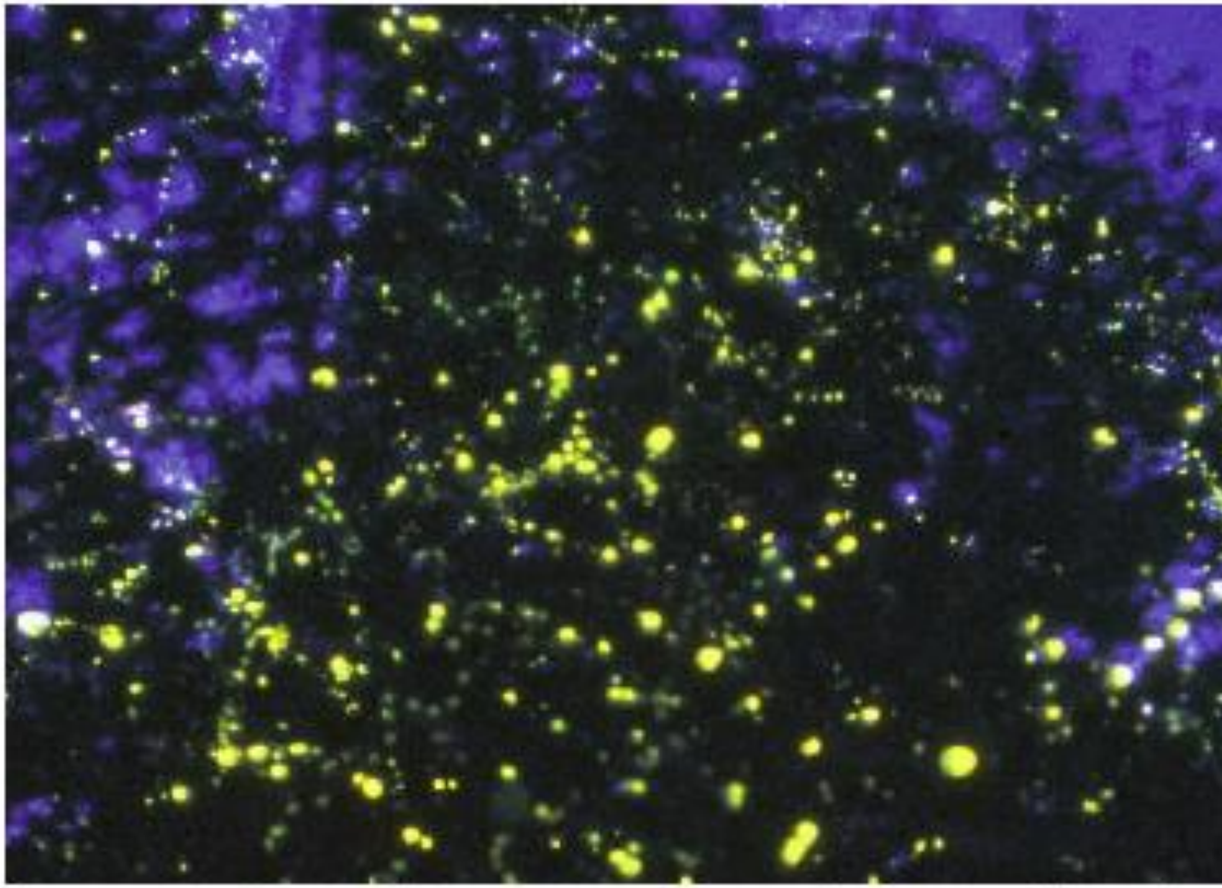


Figure 1 | Fireflies, fireflies burning bright. In the forests of the night, certain species of firefly flash in perfect synchrony — here *Pteroptyx malaccae* in a mangrove apple tree in Malaysia. Kaka *et al.*[2] and Mancoff *et al.*[3] show that the same principle can be applied to oscillators at the nanoscale.

Strogatz
video

Model (Ermentrout and Rinzel 1984)

- There is a external periodic stimulus with frequency Ω:

$$\dot{\Theta} = \Omega$$

Response of a firefly (θ) to the stimulus (Ω): if the stimulus is ahead on the firefly cycle, the firefly tries to "speed up" to synchronize;

- But if the firefly is flashing too early, then "slows down"

$$\dot{\theta} = \omega + A\sin(\Theta - \theta)$$

- If Θ is ahead θ [$0 < \Theta - \theta < \pi$] then $\sin(\Theta - \theta)>0$ and the firefly "speeds up" [$d\theta/dt > \omega$]
- The parameter *A* measures the capacity of the firefly to adapt its flashing frequency.

Analysis

$$\phi = \Theta - \theta \qquad \dot{\phi} = \dot{\Theta} - \dot{\theta} = \Omega - \omega - A\sin\phi$$

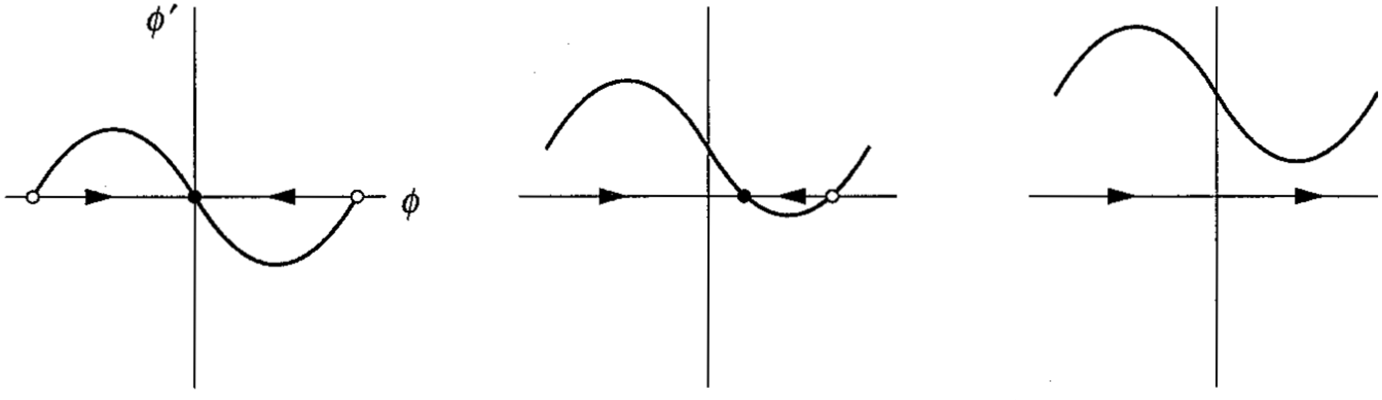
- Dimensionless model: $\tau = At \,, \quad \mu = \dfrac{\Omega - \omega}{A}$

detuning parameter

$$\phi' = \mu - \sin\phi \qquad \phi' = d\phi/d\tau$$

(Adler equation)

$$\phi = \Theta - \theta$$

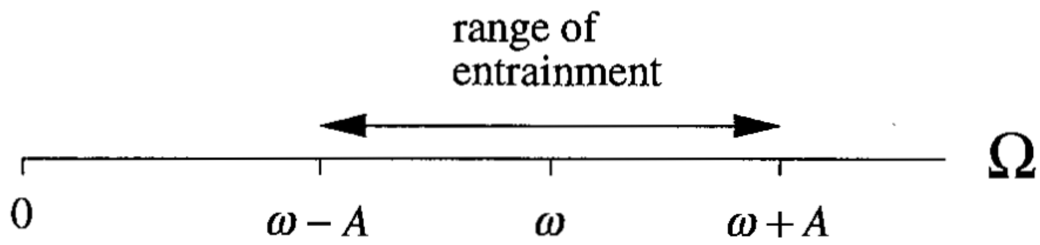


(a) $\mu = 0$ (b) $0 < \mu < 1$ (c) $\mu > 1$

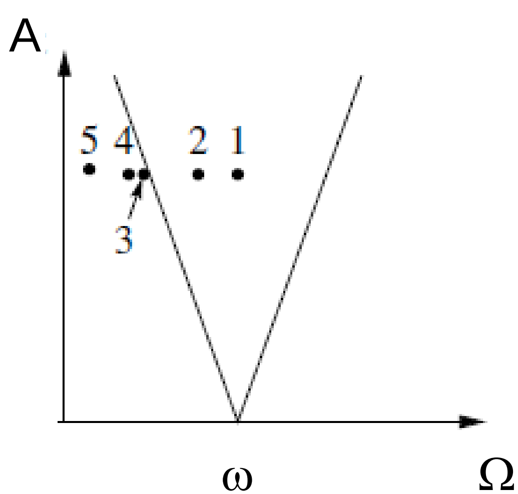
- The firefly and the stimulus flash simultaneously
- The firefly and the stimulus are **phase locked (entrainment)**: there is a stable and constant phase difference
- The firefly and the stimulus are unlocked: phase drift

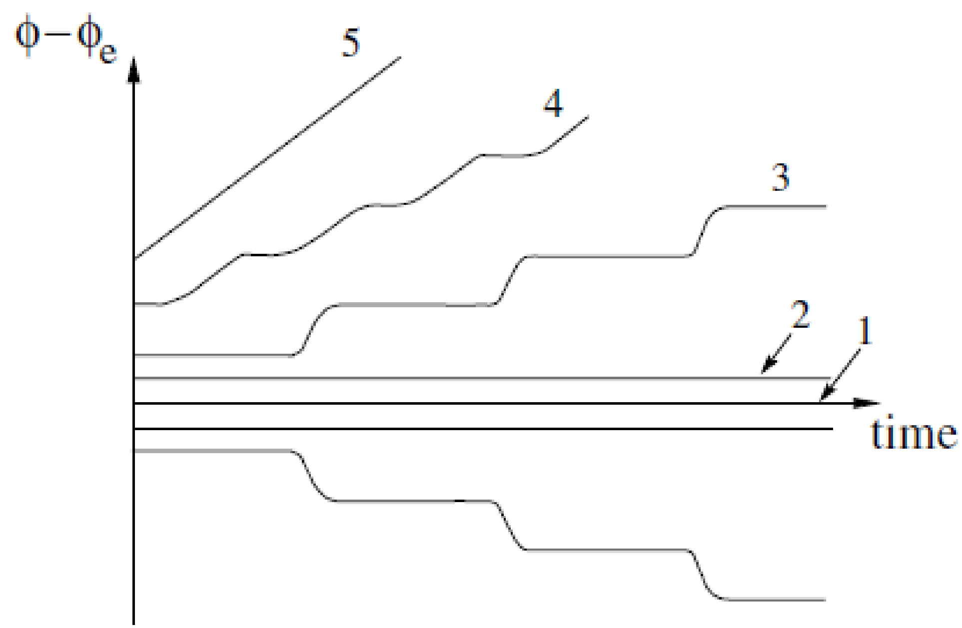
$$\phi' = \mu - \sin\phi$$

$$\mu = \frac{\Omega - \omega}{A}$$



- Entrainment is possible only if the frequency of the external stimulus, Ω, is close to the firefly frequency, ω

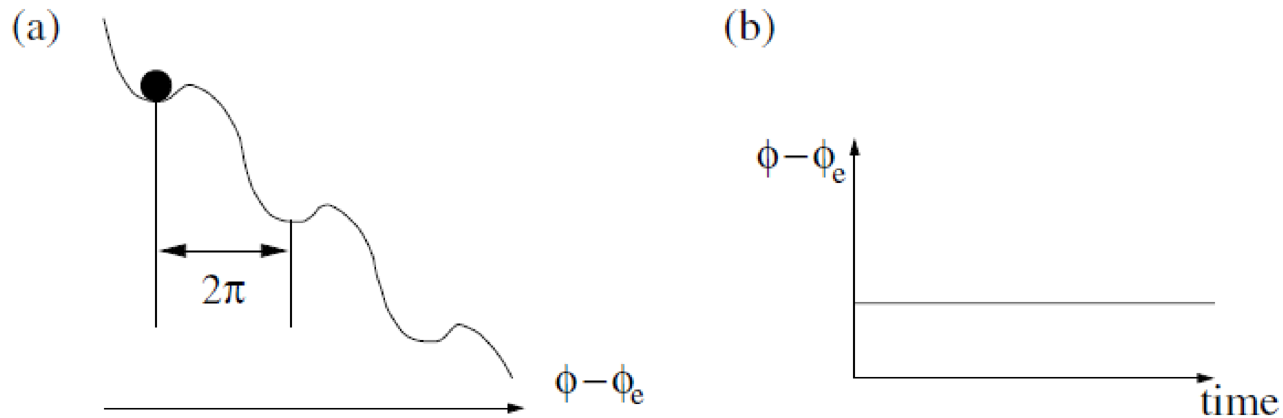




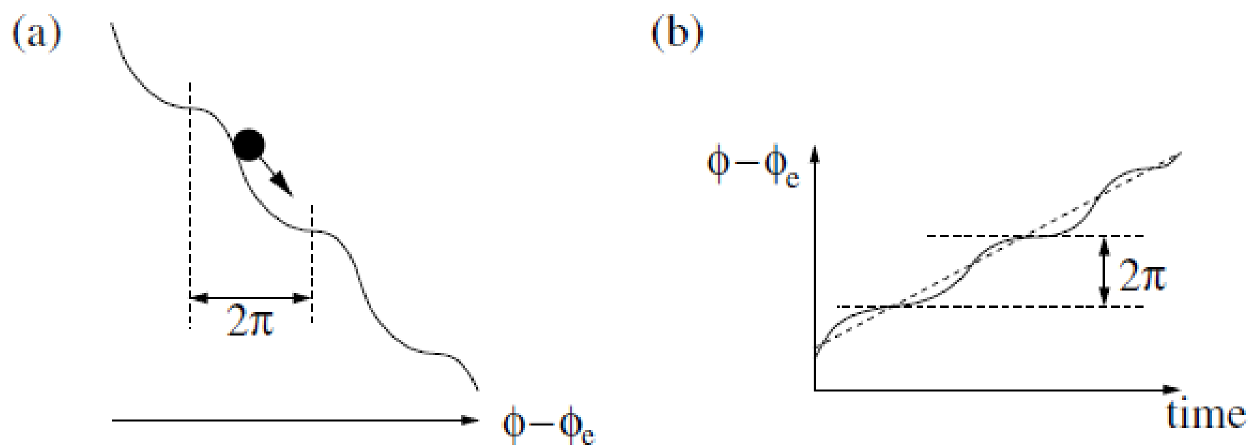
Potential interpretation

$$\phi' = \mu - \sin\phi \qquad V(\phi) = -\mu\phi - \cos\phi$$

- Small detuning



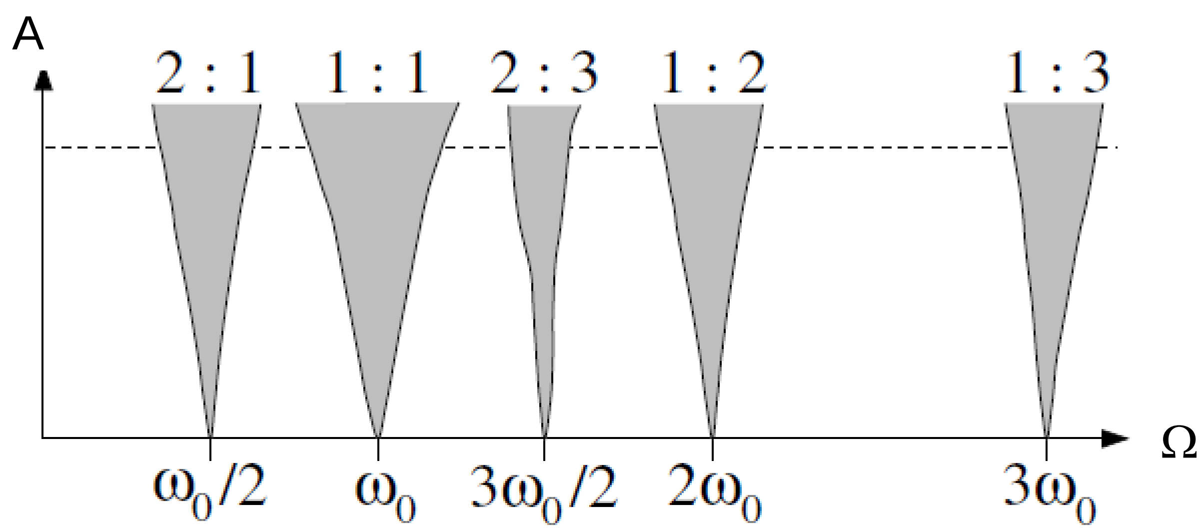
- Large detuning



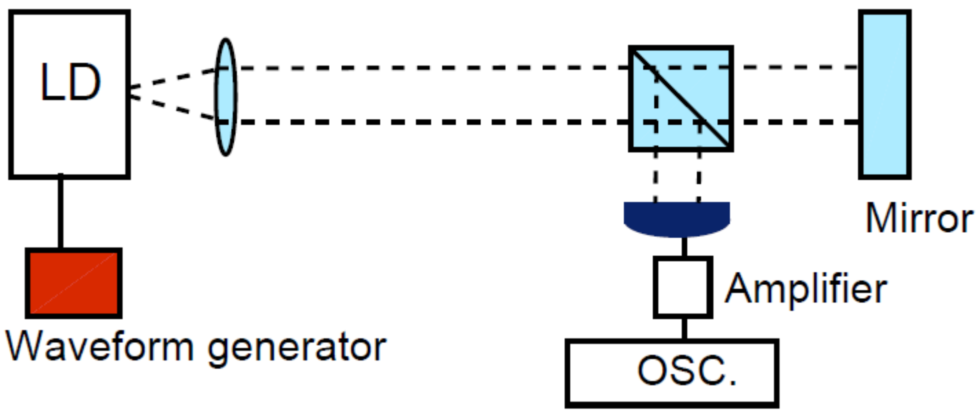
Arnold tongues

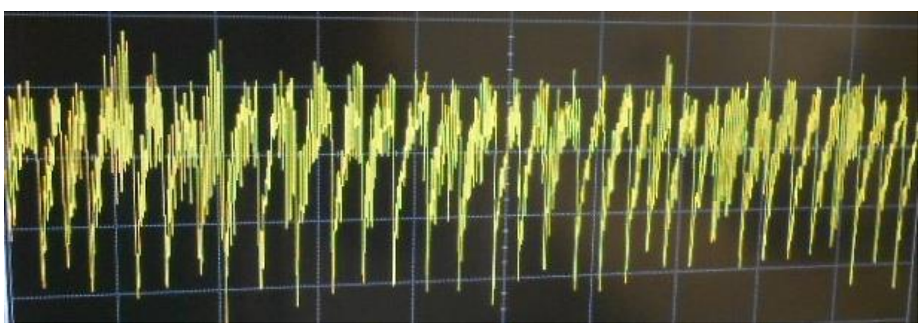
- If the external frequency Ω is not close to the firefly frequency, ω, then, a different type of synchronization is possible: the firefly can fire *m* pulses each *n* pulses of the external signal.

$$n\omega = m\Omega$$



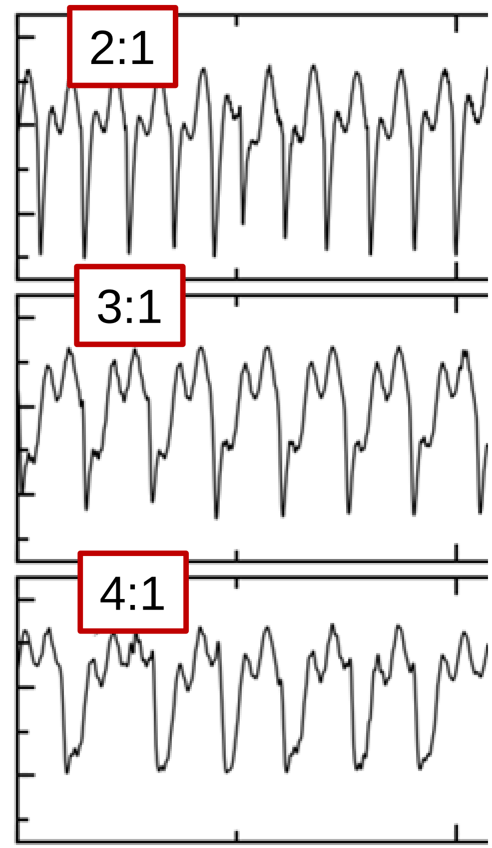
Experimental observation of entrainment





Time

$\langle T \rangle \approx 2\, T_{mod}$



Time

Summary

- A vector field on a circle is a rule that assigns a unique velocity vector to each point on the circle

$$\dot{\theta} = f(\theta) \qquad f(\theta + 2\pi) = f(\theta)$$

- $\dot{\theta} = \omega - a\sin\theta$ simple model to describe phase-locking of a nonlinear oscillator to an external periodic signal.

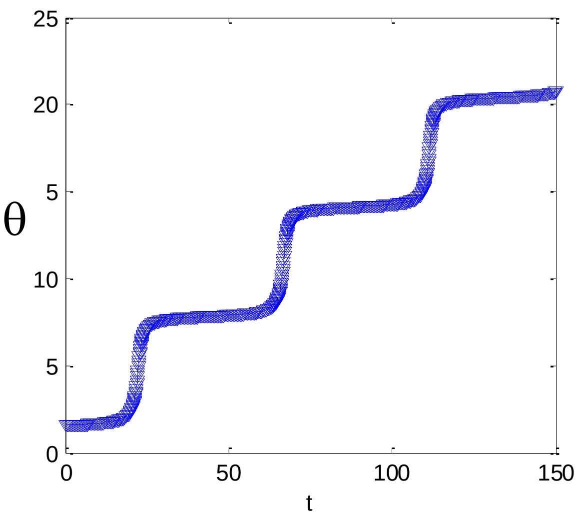
- In the phase-locked state, the oscillator maintains a constant phase difference relative to the signal.

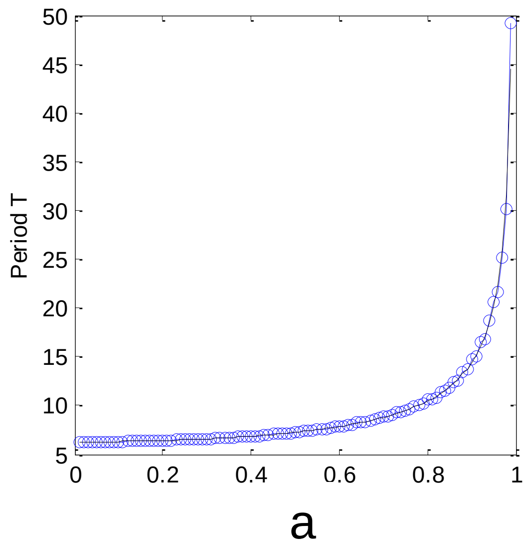
- An oscillator can be entrained to an external periodic signal if the frequencies are similar.

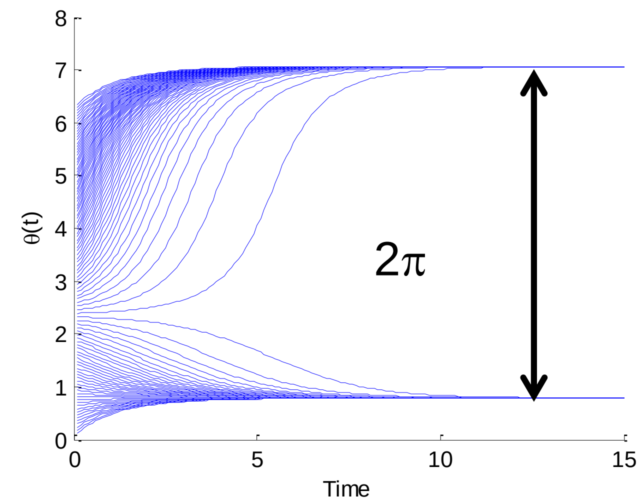
Class / Home work

$$\dot{\theta} = \omega - a\sin\theta$$

- Solve Adler's equation with $\omega=1$, $a=0.99$ and $\theta(0)=\pi/2$
- With $\omega=1$, calculate the average oscillation period and compare with the analytical expression
$$T = \frac{2\pi}{\sqrt{\omega^2 - a^2}}$$
- With $\omega=1/\mathrm{sqrt}(2)$, calculate the trajectory for an arbitrary initial condition.







Bibliography

- Steven H. Strogatz: *Nonlinear dynamics and chaos, with applications to physics, biology, chemistry and engineering* (Addison-Wesley Pub. Co., 1994). Ch. 4

- A. Pikovsky, M. Rosenblum and J. Kurths, *Synchronization, a universal concept in nonlinear science* (Cambridge University Press 2001). Chapters 1-3