Nonlinear systems, chaos and control in Engineering

Time-delayed feedback control

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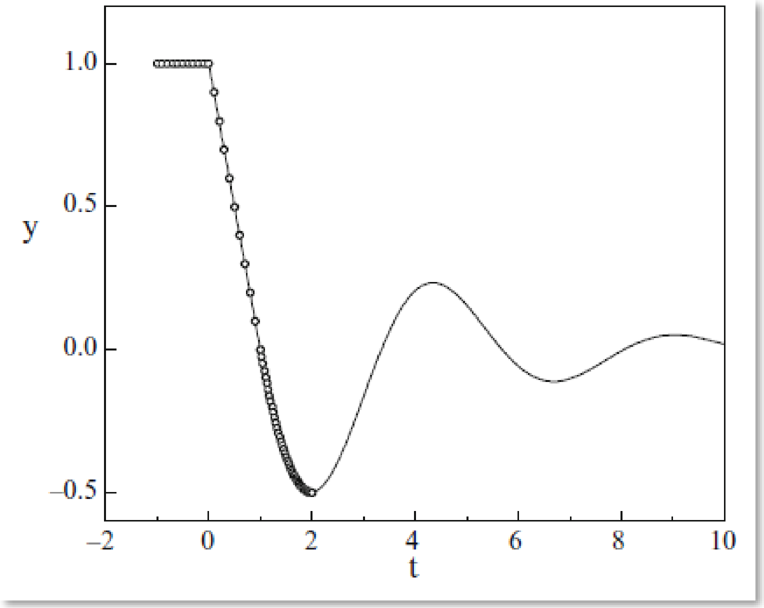
Outline

- **Introduction to delay differential equations (DDEs)**
- Solving DDEs
- Bifurcations in DDEs

Exception to no oscillations: delay differential equations (DDEs)

- Any system involving a **feedback control** will almost certainly involve time delays.
- In a 2D system delayed feedback can reduce oscillations, but in a 1D system it can induce oscillations.
- Example:

$$\frac{dy}{dt} = ky(t-\tau), \quad y(t) = 1 \quad \text{when} \quad -\tau \le t < 0$$



  - Linear system
  - Infinite-dimensional system
  - Delay-induced oscillations.

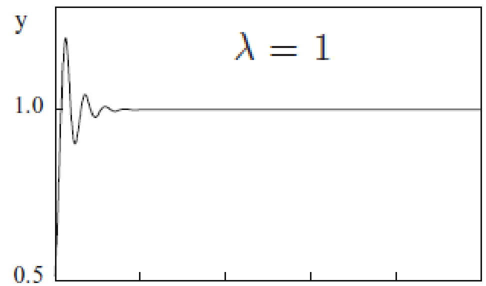
Example: population dynamics
Delayed logistic equation

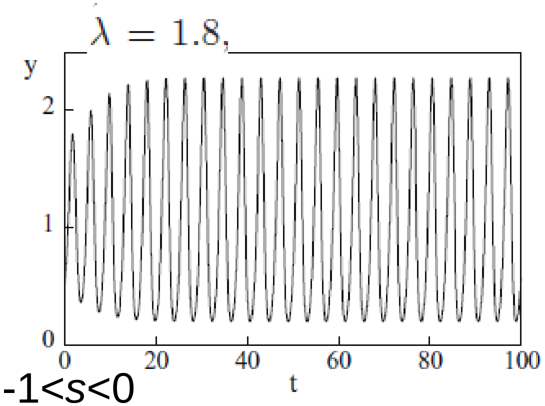
$$\frac{dN}{dt'} = rN(1 - \frac{N(t'-\tau)}{K})$$

$$y \equiv N/K \quad \text{and} \quad t \equiv t'/\tau$$

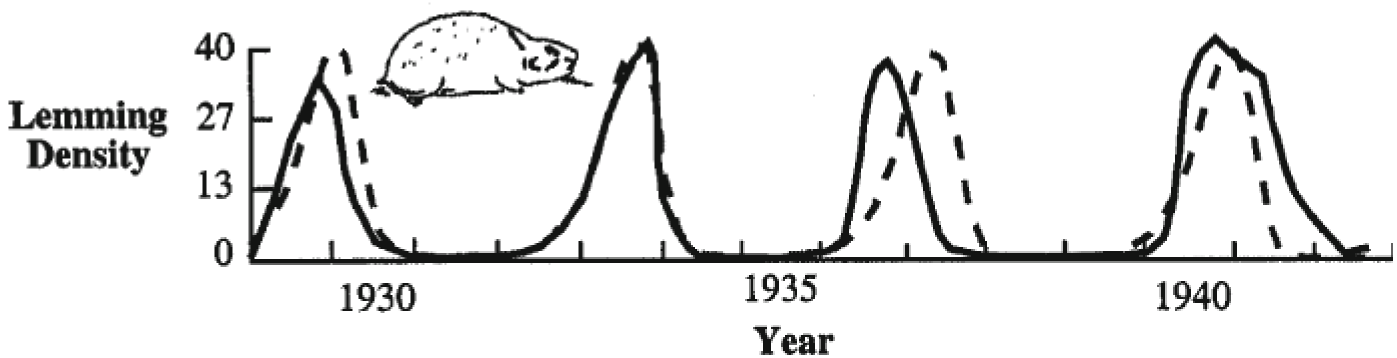
$$\frac{dy}{dt} = \lambda y(1 - y(t-1))$$

$$\lambda \equiv r\tau$$





The initial function is y=0.5 in $-1<s<0$



Lemming population cycles in the arctic north are nicely described by the logistic DDE with $r = 3.333/\text{yr}$ and $\tau = 9$ months ($\lambda = 3.333 \times 9/12 = 2.5$

- In a single-species population, the incorporation of a delay allows to explain the oscillations, without the predatory interaction of other species.

Example: Container crane
Delayed feedback control

It is important for the crane to move payloads rapidly and smoothly. If the gantry moves too fast the payload may start to sway, and it is possible for the crane operator to lose control of the payload.

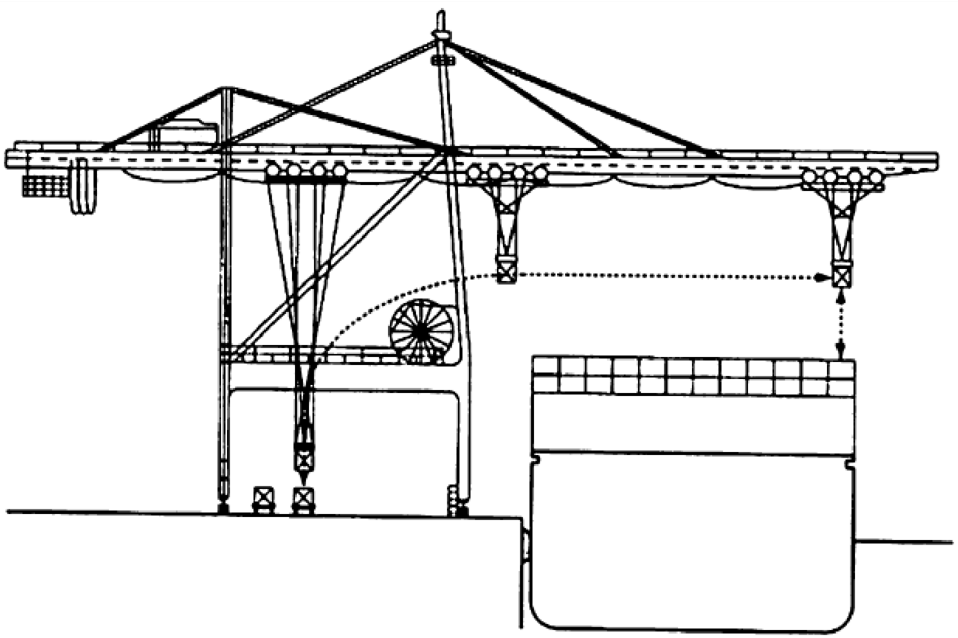


Figure 1.15: Container crane and ship (from H. Park and K.-S. Hong [181]).

Pendulum model for the crane, *y* represents the angle

$$y'' + \varepsilon y' + \sin(y) = -k\cos(y)(y(t-\tau) - y)$$

weakly damped oscillator
(not first-order equation,
without control payload
oscillations are possible)

Feedback control

Reduction of payload oscillations: why delayed feedback works?

Near the equilibrium solution *y*=0: $y'' + \varepsilon y' + y = -k(y(t-\tau) - y)$

Small delay: $y(t-\tau) \sim y - \tau y'$

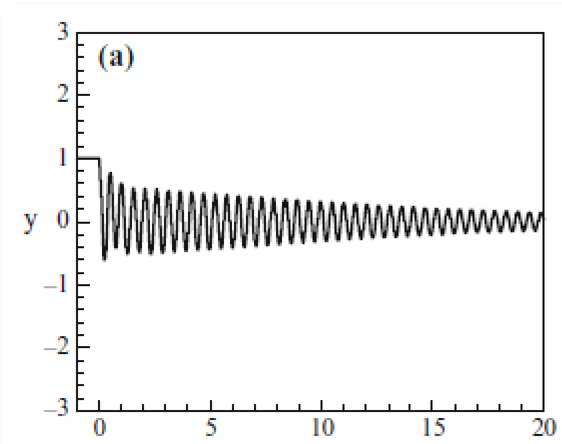
$$y'' + (\varepsilon - k\tau)y' + y = 0$$

The delay increases the damping.

Therefore: the oscillations decay faster.

but

- Small perturbation



- Large perturbation

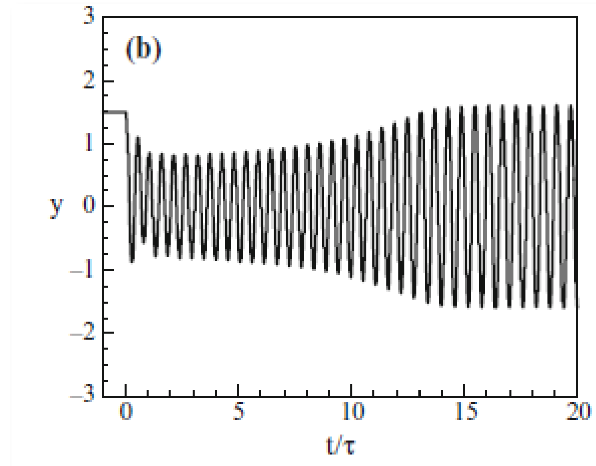


Figure 1.16: The values of the fixed parameters are $\tau = 12$, $\varepsilon = 0.1$, and $k = -0.15$. (a): $y'(0) = 0$ and $y = 1$ $(-\tau < t < 0)$; (b): $y'(0) = 0$ and $y = 1.5$ $(-\tau < t < 0)$.

Example: Car following model

$$x''_{n+1}(t+\tau) = \alpha(x'_n - x'_{n+1})$$

can be used for determining the location and speed of the following car (at $x = x_{n+1}$) given the speed pattern of the leading vehicle (at $x = x_n$). If a driver reacts too strongly (large value of α representing excessive braking) or too late (long reaction time τ), the spacing between vehicles may become unstable (i.e., we note damped oscillations in the spacing between vehicles).

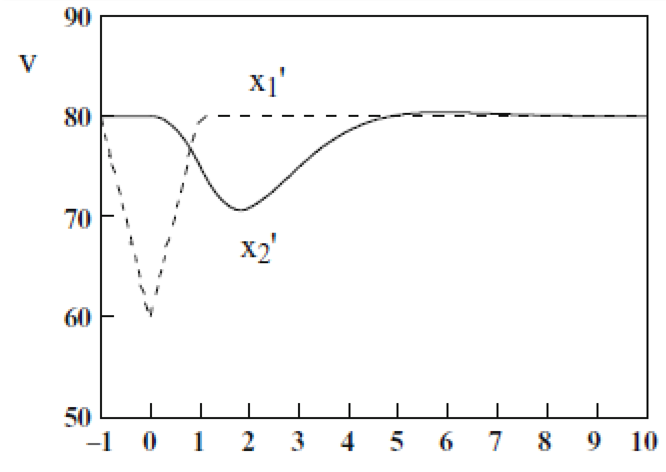
Typical solution for two cars

$$x''_{n+1}(t+\tau) = \alpha(x'_n - x'_{n+1})$$

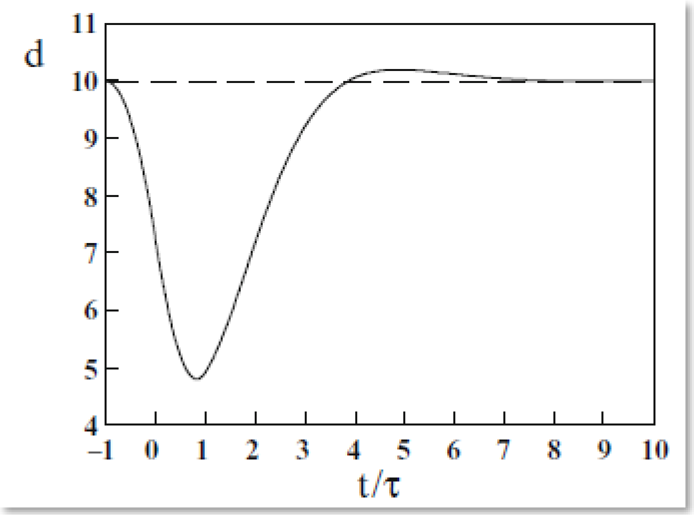
The lead vehicle reduces its speed of 80 km/h to 60 km/h and then accelerates back to its original speed. The initial spacing between vehicles is 10 m.

$\alpha = 0.5\ \mathrm{s}^{-1}$ and $\tau = 1\ \mathrm{s}$

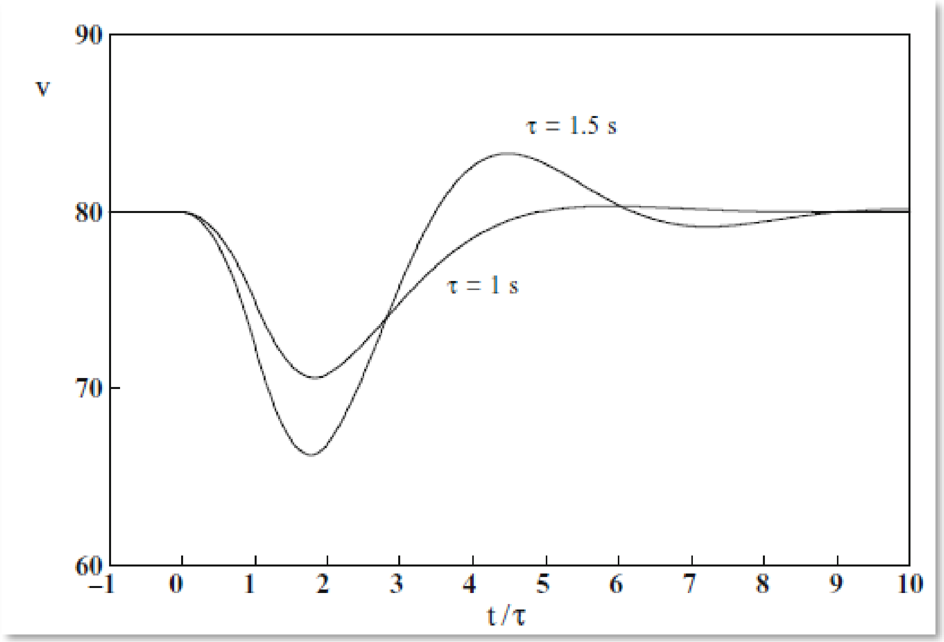
- Speed of the two cars



- Distance between the two cars



Alcohol effect



A sober driver needs about 1 s in order to start breaking in view of an obstacle.
With 0.5 g/l alcohol in blood (2 glasses of wine), this reaction time is estimated to be about 1.5 s.
$\Rightarrow$ oscillations near the stable equilibrium increase.

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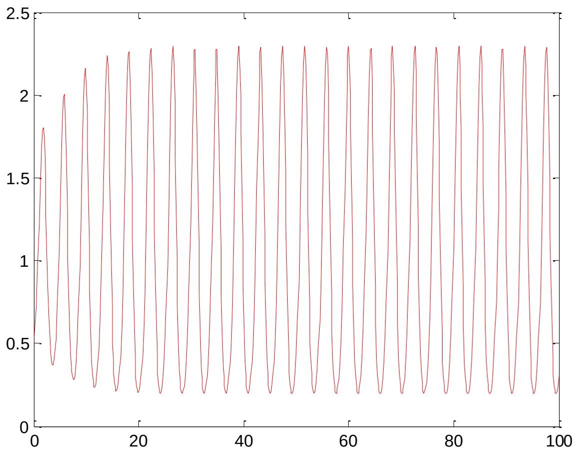
- **Solving DDEs**
- Bifurcations in DDEs

Example 1: Delayed logistic equation

$$\frac{dy}{dt} = \lambda y(1 - y(t-1))$$

ic =constant initial function

```
function solve_delay1
tau = 1;
ic = [0.5];
tspan = [0 100];
h = 1.8;
sol = dde23(@f,tau,ic,tspan);
plot(sol.x,sol.y(1,:),'r-')
function v=f(t,y,Z)
v = [h*y(1).*(1-Z(1))];
end
end
```

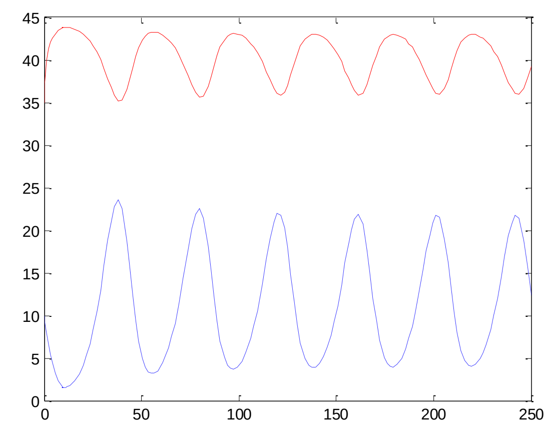


Solving DDEs

Example 2: Prey (x) and predator (y) model

$$\frac{dx}{dt} = x(t)\left\{2\left[1-\frac{x(t)}{50}\right]-\frac{y(t)}{x(t)+40}\right\}-10,$$
$$\frac{dy}{dt} = y(t)\left[-3+\frac{6x(t-\tau)}{x(t-\tau)+40}\right].$$

```
function solve_delay2
tau=9;
ic = [35;10];
tspan = [0 250];
h = 10;
sol = dde23(@f,tau,ic,tspan);
plot(sol.x,sol.y(1,:),'r-',sol.x,sol.y(2,:),'b--')
function v=f(t,y,Z)
v = [y(1)*(2*(1-y(1)/50)-y(2)/(y(1)+40))-h
    y(2)*(-3+6*Z(1)/(Z(1)+40))];
end
end
```



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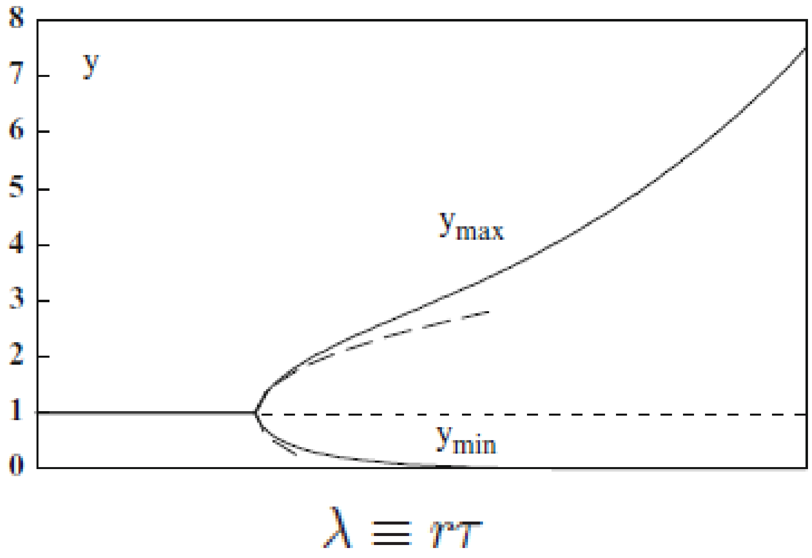
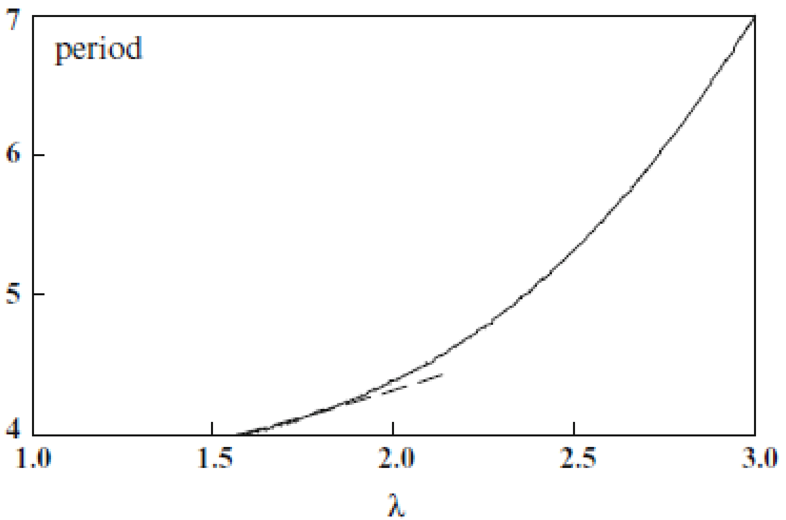
- **Bifurcations in DDEs**

Bifurcations in 1D systems with delay
Example 1: delayed logistic equation

$$\frac{dN}{dt'} = rN(1 - \frac{N(t'-\tau)}{K}) \qquad \frac{dy}{dt} = \lambda y(1 - y(t-1)) \qquad \lambda \equiv r\tau$$

- Delay allows for sustained oscillations in a single species population, without any predatory interaction of other species





- Hopf bifurcation

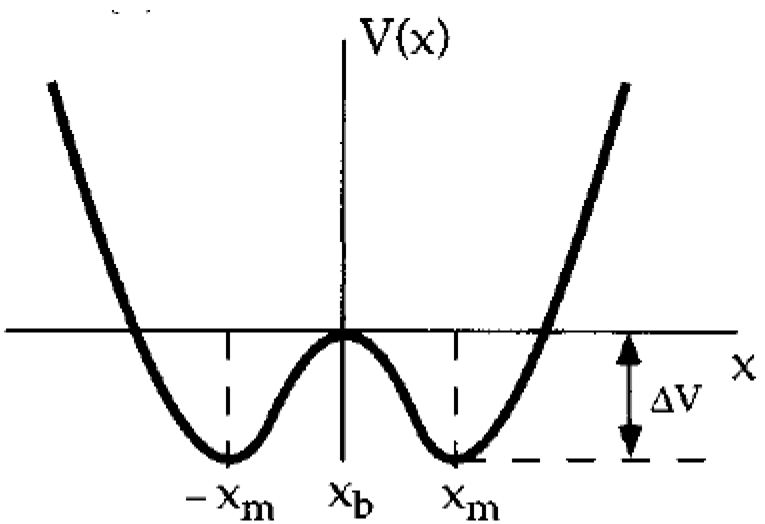
Example 2: particle in a double-well potential with delayed feedback

$$\frac{dx}{dt} = x - x^3 + c\,x(t-\tau) + \sqrt{2D}\xi$$

noise

$$= -\frac{\partial V}{\partial x} + c\,x(t-\tau) + \sqrt{2D}\xi$$

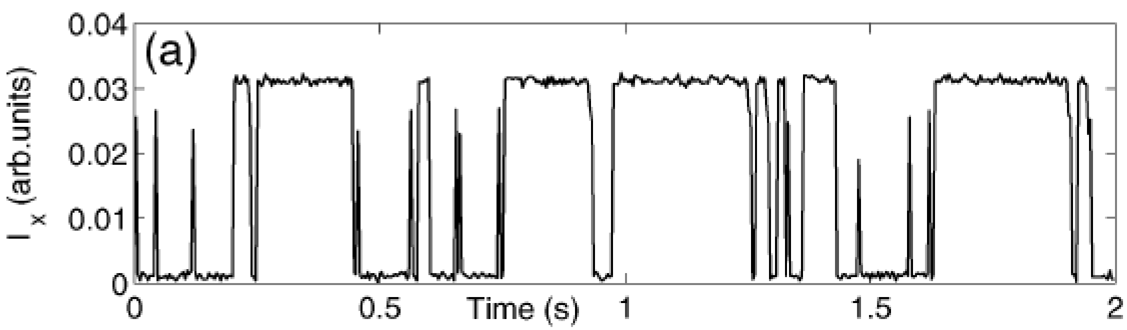
$$V = -x^2/2 + x^4/4$$



Motivation

$$\frac{dx}{dt} = x - x^3 + c\,x(t-\tau) + \sqrt{2D}\xi$$

- Simple model to understand two-state systems
- Example: in the light emitted by a laser with feedback, observation of switching between X and Y polarization.



- With appropriated parameters delay feedback can control the movement and confine the system in one state.

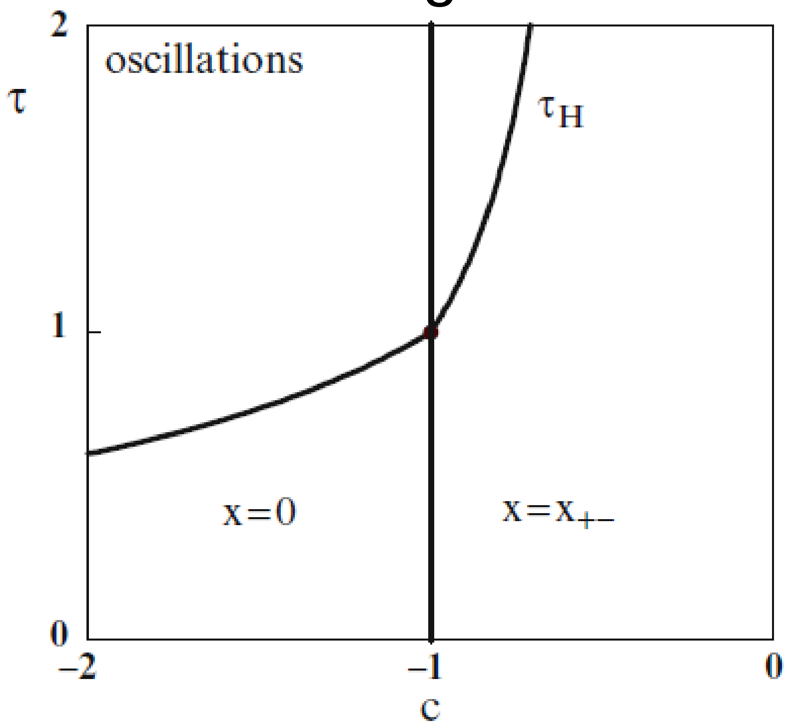
Fixed point solutions (deterministic equation, D=0)

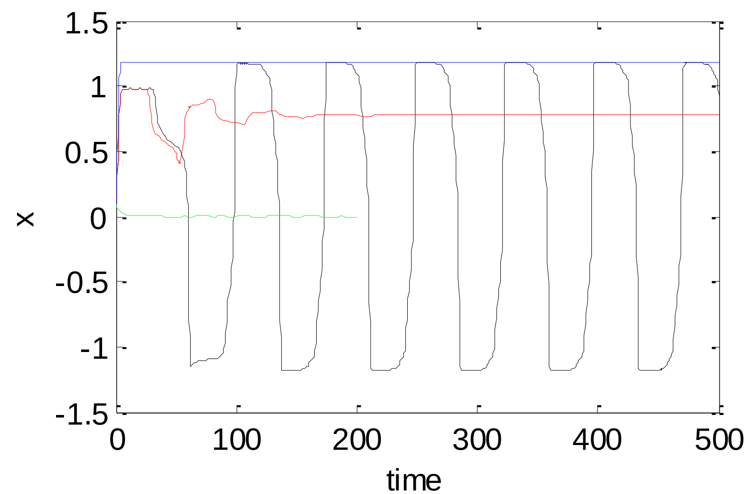
$$x = 0,$$

$$x = x_\pm \equiv \pm\sqrt{1+c} \;\; (c \geq -1)$$

if c>0 stable for all τ
if c<0 the stability depends on (c, τ)

- Phase Diagram





blue: c=0.4, τ=1
red: c=-0.4, τ=25
black: c=-0.4, τ=30
green: c=-1.1, τ=0.5

Special initial conditions give meta-stability and long transients

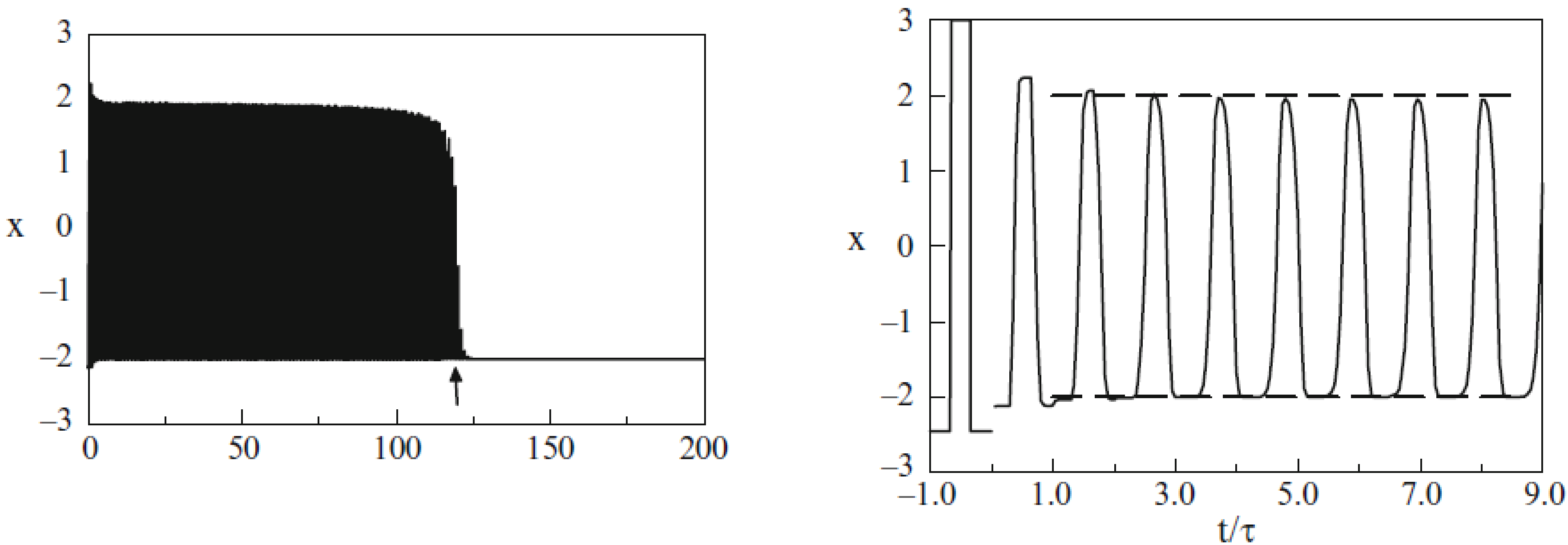


Figure 2.10: Top: slowly varying oscillations followed by a sudden jump to the steady-state $x = -2$. Bottom: short time solution showing the initial conditions: $x = -2.45$ ($-\tau < t < -2\tau/3$ and $-\tau/3 < t < 0$) and $x = 3$ ($-2\tau/3 < t < -\tau/3$). The values of the parameters are $c = 3$ and $\tau = 5$.

Bibliography

- Thomas Erneux: *Applied delay differential equations* (Springer 2009).

- D. J. Higham and N. J. Higham, *Matlab Guide Second Edition* (SIAM 2005)