Nonlinear systems, chaos and control in Engineering

Module 1
One-dimensional nonlinear systems

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Flows on the line = first-order ordinary differential equations.

\[ \frac{dx}{dt} = f(x) \]

Fixed point solutions: \( f(x^*) = 0 \)
- stable if \( f'(x^*) < 0 \)
- unstable if \( f'(x^*) > 0 \)
- neutral (bifurcation point) if \( f'(x^*) = 0 \)

There are no periodic solutions; the approach to fixed point solutions is monotonic (sigmoidal or exponential).

In a 2D system a delayed control term can reduce oscillations; in a 1D system it can induce oscillations.
Introduction to bifurcations

Saddle-node, transcritical and pitchfork bifurcations

Examples
  • Saddle-node: neuron model
  • Transcritical: laser threshold
  • Pitchfork: particle in a rotating wire hoop

Imperfect bifurcations & catastrophes
  • Example: insect outbreak

Delayed feedback examples
We consider a system with a control parameter.

Bifurcation: a qualitative change (in the structure of the phase space) when the control parameter is varied:

- Fixed points can be created or destroyed
- The stability of a fixed point can change

There are many examples in physical systems, biological systems, etc.
Control parameter increases in time
Bifurcation and potential

Monostability

Bifurcation

Bistability
Bifurcations are not equivalent to qualitative change of behavior

Bifurcation but no change of behavior

Change of behavior but no bifurcation
Outline

- Introduction to bifurcations
- Saddle-node, transcritical and pitchfork bifurcations
- Examples
  - Saddle-node: neuron model
  - Transcritical: laser threshold
  - Pitchfork: particle in a rotating wire hoop
- Imperfect bifurcations & catastrophes
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- Delayed feedback examples
Saddle-node bifurcation

Basic mechanism for the creation or the destruction of fixed points

\[ \dot{x} = f(x) = r + x^2 \quad \text{or} \quad x^* = \pm \sqrt{-r} \]

(a) \( r < 0 \)

\[ f'(-\sqrt{-r}) = -2\sqrt{-r} \quad \text{Stable if } r < 0 \]

\[ f'(-\sqrt{-r}) = 2\sqrt{-r} \quad \text{unstable} \]

(b) \( r = 0 \)

At the bifurcation point \( r^* = 0 \): \( f'(x^*) = 0 \)
\[
\dot{x} = r - x^2
\]

- Calculate the fixed points and their stability as a function of the control parameter \( r \)

\[
x^* = \pm \sqrt{r}
\]

- For \( r < 0 \)
- For \( r = 0 \)
- For \( r > 0 \)
- Are representative of all saddle-node bifurcations.

- Close to the saddle-node bifurcation the dynamics can be approximated by

\[
\dot{x} = r - x^2 \quad \text{or} \quad \dot{x} = r + x^2
\]

Example:

\[
\dot{x} = r - x - e^{-x} \\
= r - x - \left[ 1 - x + \frac{x^2}{2} + \cdots \right] \\
= (r - 1) - \frac{x^2}{2} + \cdots
\]
Near a saddle-node bifurcation

$f(x)$ looks parabolic in here
A pair of fixed points appear (or disappear) out of the “clear blue sky” ("blue sky" bifurcation, Abraham and Shaw 1988).
Transcritical bifurcation: general mechanism for changing the stability of fixed points.

\[ \dot{x} = rx - x^2 \]

\[ x^* = 0 \]
\[ x^* = r \]

are the fixed points for all \( r \)

Transcritical bifurcation: general mechanism for changing the stability of fixed points.
\[ \dot{x} = rx - x^2 \]

fixed points \( x^* = 0 \) and \( x^* = r \)

\[ f'(x) = r - 2x \]

\[ f'(0) = r \]

\[ f'(r) = -r \]

- Exchange of stability at \( r = 0 \).

- **Exercise:** \( \dot{x} = r \ln x + x - 1 \)
  
  show that a transcritical bifurcation occurs near \( x=1 \)
  
  (hint: consider \( u = x-1 \) small)
Pitchfork bifurcation

\[ \dot{x} = rx - x^3 \]

Symmetry \( x \rightarrow -x \)

(a) \( r < 0 \)

(b) \( r = 0 \)

(c) \( r > 0 \)

One fixed point → 3 fixed points
The governing equation is symmetric: $x \rightarrow -x$ but for $r > 0$: symmetry broken solutions.
\[
\dot{x} = rx - x^3 \quad \dot{x} = f(x) \quad f(x) = -\frac{dV}{dx} \quad V(x) = -\frac{1}{2} rx^2 + \frac{1}{4} x^4
\]
Pitchfork bifurcations

Supercritical: $x^3$ is stabilizing

$$\dot{x} = rx - x^3$$

Subcritical: $x^3$ is destabilizing

$$\dot{x} = rx + x^3$$
Exercise: find the fixed points and compute their stability

\[ \dot{x} = rx + x^3 - x^5 \]
Critical or dangerous transition! A lot of effort in trying to find “early warning signals” (more latter)
Hysteresis: sudden changes in visual perception

Fischer (1967): experiment with 57 students.

“When do you notice an abrupt change in perception?”
Bifurcation condition: change in the stability of a fixed point

\[ f'(x^*) = 0 \]

In first-order ODEs: three possible bifurcations
- Saddle node
- Pitchfork
- Trans-critical

The normal form describes the behavior near the bifurcation.
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Imperfect bifurcations & catastrophes
  • Example: insect outbreak
Delayed feedback examples
Example: neuron model

\[
C \frac{dV}{dt} = I - g_L(V - E_L) - g_{Na} m_{\infty}(V)(V - E_{Na})
\]

\[
m_{\infty}(V) = \frac{1}{1 + \exp\left\{\left(V_{1/2} - V\right)/k\right\}}
\]

\[
C = 10 \mu F, \quad I = 0 \text{ pA}, \quad g_L = 19 \text{ mS}, \quad E_L = -67 \text{ mV},
\]
\[
g_{Na} = 74 \text{ mS}, \quad V_{1/2} = 1.5 \text{ mV}, \quad k = 16 \text{ mV}, \quad E_{Na} = 60 \text{ mV}
\]

I = 0

bistability

\[F(V)\]

rest, threshold

\[V(t)\]

membrane potential, V (mV)

excited state

threshold

rest
Bifurcation

For $I=16$, the system shows a clear bifurcation point where the membrane potential $V(t)$ transitions to an excited state.

When $I=60$, the system exhibits monostability, with a single stable state for the membrane potential $V(t)$ over time $t$. The diagrams illustrate the changes in membrane potential with respect to current $I$ and time $t$. The tangent point and the excited state are marked for each condition.
Near the bifurcation point: slow dynamics

This slow transition is an “early warning signal” of a critical or dangerous transition ahead (more latter)
If the control parameter now decreases
Simulate the neuron model with different values of the control parameter I and/or different initial initial conditions.
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Delayed feedback examples
\[ \dot{n} = \text{gain} - \text{loss} = G n N - k n. \]

\[ N(t) = N_0 - \alpha n \]

\[ \dot{n} = G n (N_0 - \alpha n) - k n = (G N_0 - k) n - (\alpha G) n^2. \]

\[ \dot{x} = r x - x^2 \]

\[ N_0 < k/G \quad \quad \quad \quad N_0 = k/G \quad \quad \quad \quad N_0 > k/G \]
Bifurcation diagram: LI curve
\[ \dot{x} = rx - x^2 \]

\[ r(t) = r_0 + vt \]

Linear increase of control parameter

\[ r_0 < r^* = 0 \]

Start before the bifurcation point
Comparison with experimental observations

**Dynamical hysteresis**

Quasi-static very slow variation of the control parameter

Am. J. Phys., Vol. 72, No. 6, June 2004
Laser turn-on

\[ r(t) = r_0 \quad r_0 > r^* = 0 \]

Fig. 1.3 He-Ne gas laser output as a function of time. From the lower to the upper time traces, the pump parameter above threshold is gradually increased. Reprinted Figure 2 with permission from Pariser and Marshall [30]. Copyright 1965 by the American Institute of Physics.
We need more equations to explain these oscillations. The diode laser is not a 1D system.
With “noise”:
“imperfect” bifurcation

\[ \dot{x} = f(x) + \xi(t) \]

\[ \xi = 0 \]

\[ \xi \neq 0 \]

\[ \dot{n} = \text{gain} - \text{loss} + \beta \]

Fig. 1.17 Imperfect bifurcation for a laser in the presence of spontaneous emission, measured for a He-Ne laser. Reprinted Figure 1 with permission from Corti and Degiorgio [42]. Copyright 1976 by the American Physical Society.
Simulate the laser model when the control parameter $r$ increases linearly in time. Consider different variation rate ($v$) and/or different initial value of the parameter ($r_0$).

\[ r(t) = r_0 + vt \]

\[ x_0 = 0.01 \]

\[ v = 0.1 \]

\[ x_0 = 0.01 \]
Now consider that the control parameter $r$ increases and then decreases linearly in time.

Plot $x$ and $r$ vs time and plot $x$ vs $r$. 

$v = 0.01$
Calculate the “turn on” when $r$ is constant, $r > r^* = 0$.

\[ r(t) = r \]

\[ x_0 = 0.01 \]

Calculate the bifurcation diagram by plotting $x(t=50)$ vs $r$.

\[ \dot{x} = rx - x^2 + h \]
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Delayed feedback examples
A particle moves along a wire hoop that rotates at constant angular velocity.

\[
m r \ddot{\phi} = -b \dot{\phi} - mg \sin \phi + m r \omega^2 \sin \phi \cos \phi
\]
Neglect the second derivative (more latter)

\[ b\dot{\phi} = -mg \sin \phi + mr\omega^2 \sin \phi \cos \phi \]

\[ = mg \sin \phi \left( \frac{r\omega^2}{g} \cos \phi - 1 \right) \]

- Fixed points from: \( \sin \phi = 0 \)

\( \phi^* = 0 \) (the bottom of the hoop) and \( \phi^* = \pi \) (the top)

- Fixed points from: \( \gamma \cos \phi - 1 = 0 \)

\[ \gamma = \frac{r\omega^2}{g} \]
When is this “first-order” description valid? When is ok to neglect the second derivative $d^2x/dt^2$?

Dimensional analysis and scaling
- Dimensionless time

\( T = \text{characteristic time-scale} \)

\[
\left( \frac{r}{gT^2} \right) \frac{d^2 \phi}{d\tau^2} = -\left( \frac{b}{mgT} \right) \frac{d\phi}{d\tau} - \sin\phi + \left( \frac{r\omega^2}{g} \right) \sin\phi \cos\phi
\]

- We want the lhs very small, we define \( T \) such that

\[
\frac{r}{gT^2} \ll 1 \quad \text{and} \quad \frac{b}{mgT} \approx O(1) \quad \Rightarrow \quad T = \frac{b}{mg}
\]

\[
\frac{r}{g} \left( \frac{mg}{b} \right)^2 \ll 1 \quad \Rightarrow \quad b^2 >> m^2 gr
\]

- Define: \( \varepsilon = \frac{m^2 gr}{b^2} \)

\[
\varepsilon \frac{d^2 \phi}{d\tau^2} = -\frac{d\phi}{d\tau} - \sin\phi + \gamma \sin\phi \cos\phi
\]

\[
\gamma = \frac{r\omega^2}{g}
\]
The dimension less equation suggests that the first-order equation is valid in the over damped limit: $\epsilon \to 0$

Problem: second-order equation has two independent initial conditions: $\phi(0)$ and $\frac{d\phi}{d\tau}(0)$

But the first-order equation has only one initial condition $\phi(0)$, $\frac{d\phi}{d\tau}(0)$ is calculated from

$$\frac{d\phi}{d\tau} = -\sin \phi + \gamma \sin \phi \cos \phi$$

Paradox: how can the first-order equation represent the second-order equation?
Trajectories in phase space

- **First order system:**

\[
\frac{d\phi}{d\tau} = f(\phi) - \sin \phi + \gamma \sin \phi \cos \phi
\]

\[
\Omega = \phi' \equiv \frac{d\phi}{d\tau}
\]

![Trajectory diagram](image)

- **Second order system:**

\[
\varepsilon \frac{d^2\phi}{d\tau^2} = -\frac{d\phi}{d\tau} - \sin \phi + \gamma \sin \phi \cos \phi
\]

\[
C: f(\phi) - \Omega = 0
\]

![Trajectory diagram](image)

Second order system:

\(\varepsilon \to 0\)

*limit, all trajectories slam straight up or down onto the curve C defined by \(f(\phi) = \Omega,\) and then slowly ooze along this curve until they reach a fixed point*
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Delayed feedback examples
Imperfect bifurcations

\[ \dot{x} = h + rx - x^3 \]

(a) \( h = 0 \)

(b) \( h \neq 0 \)
Exercise: using these two equations

1. fixed points: \( f(x^*) = 0 \)
2. saddle node bifurcation: \( f'(x^*) = 0 \)

Calculate \( h_c(r) \)

\[
\hat{x} = h + rx - x^3
\]
Example: insect outbreak

\[ \dot{N} = RN \left(1 - \frac{N}{K}\right) - p(N) \]

- Budworms population grows logistically \((R>0 \text{ grow rate})\)
- \(p(N)\): dead rate due to predation
- If no budworms \((N \approx 0)\): no predation: birds look for food elsewhere
- If \(N\) large, \(p(N)\) saturates: birds eat as much as they can.

\[ p(N) = \frac{BN^2}{A^2 + N^2} \quad A, B > 0 \]
Dimensionless formulation

\[ x = \frac{N}{A} \quad \tau = \frac{Bt}{A} \quad r = \frac{RA}{B} \quad k = \frac{K}{A} \]

\[ \frac{dx}{d\tau} = r x \left( 1 - \frac{x}{k} \right) - \frac{x^2}{1 + x^2} \]

- \( x^* = 0 \)
- Other FPs from the solution of \( r \left( 1 - \frac{x}{k} \right) = \frac{x}{1 + x^2} \) are independent of \( r \) and \( k \)
- When the line intersects the curve tangentially (dashed line): saddle-node bifurcation

Exercise: show that $x^*=0$ is always unstable
Parameter space \((k, r)\)
- Introduction to bifurcations
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Bifurcations in 1D systems with delay

Example 1: delayed logistic equation

\[ \frac{dN}{dt'} = rN\left(1 - \frac{N(t' - \tau)}{K}\right) \]
\[ \frac{dy}{dt} = \lambda y(1 - y(t - 1)) \]

- Delay allows for sustained oscillations in a single species population, without any predatory interaction of other species.

- Hopf bifurcation (more in Part 2 = 2D dynamical systems)
Example 2: particle in a double-well potential with delayed feedback

\[ \frac{dx}{dt} = x - x^3 + c \, x(t-\tau) + \sqrt{2D} \xi \]

noise

\[ = -\frac{\partial V}{\partial x} + c \, x(t-\tau) + \sqrt{2D} \xi \]

\[ V = -x^2 / 2 + x^4 / 4 \]
Simple model to understand two-state systems

Example: in the light emitted by a laser with feedback, observation of switching between X and Y polarization.

With appropriated parameters delay feedback can control the movement and confine the system in one state.
Fixed point solutions
(deterministic equation, D=0)

\[ x = 0, \]
\[ x = x_\pm = \pm \sqrt{1 + c} \quad (c \geq -1) \]

- if \( c > 0 \) stable for all \( \tau \)
- if \( c < 0 \) the stability depends on \((c, \tau)\)

Phase Diagram

- blue: \( c=0.4, \ \tau=1 \)
- red: \( c=-0.4, \ \tau=25 \)
- black: \( c=-0.4, \ \tau=30 \)
- green: \( c=-1.1, \ \tau=0.5 \)
Special initial conditions give meta-stability and long transients.

Figure 2.10: Top: slowly varying oscillations followed by a sudden jump to the steady-state $x = -2$. Bottom: short time solution showing the initial conditions: $x = -2.45$ ($-\tau < t < -2\tau/3$ and $-\tau/3 < t < 0$) and $x = 3$ ($-2\tau/3 < t < -\tau/3$). The values of the parameters are $c = 3$ and $\tau = 5$. 
3.5.4 (Bead on a horizontal wire) A bead of mass $m$ is constrained to slide along a straight horizontal wire. A spring of relaxed length $L_0$ and spring constant $k$ is attached to the mass and to a support point a distance $h$ from the wire (Figure 1).

![Diagram of a bead on a horizontal wire with a spring](image)

**Figure 1**

Finally, suppose that the motion of the bead is opposed by a viscous damping force $b\dot{x}$.

a) Write Newton's law for the motion of the bead.
b) Find all possible equilibria, i.e., fixed points, as functions of $k$, $h$, $m$, $b$, and $L_0$.
c) Suppose $m = 0$. Classify the stability of all the fixed points, and draw a bifurcation diagram.
d) If $m \neq 0$, how small does $m$ have to be to be considered negligible? In what sense is it negligible?
3.7.6 (Model of an epidemic) In pioneering work in epidemiology, Kermack and McKendrick (1927) proposed the following simple model for the evolution of an epidemic. Suppose that the population can be divided into three classes: $x(t) =$ number of healthy people; $y(t) =$ number of sick people; $z(t) =$ number of dead people. Assume that the total population remains constant in size, except for deaths due to the epidemic. (That is, the epidemic evolves so rapidly that we can ignore the slower changes in the populations due to births, emigration, or deaths by other causes.)

Then the model is

$$
\begin{align*}
\dot{x} &= -kxy \\
\dot{y} &= kxy - \ell y \\
\dot{z} &= \ell y
\end{align*}
$$

where $k$ and $\ell$ are positive constants. The equations are based on two assumptions:

(i) Healthy people get sick at a rate proportional to the product of $x$ and $y$. This would be true if healthy and sick people encounter each other at a rate proportional to their numbers, and if there were a constant probability that each such encounter would lead to transmission of the disease.

(ii) Sick people die at a constant rate $\ell$.

The goal of this exercise is to reduce the model, which is a third-order system, to a first-order system that can analyzed by our methods. (In Chapter 6 we will see
a) Show that \( x + y + z = N \), where \( N \) is constant.

b) Use the \( \dot{x} \) and \( \dot{z} \) equation to show that \( x(t) = x_0 \exp(-kz(t)/\ell) \), where \( x_0 = x(0) \).

c) Show that \( z \) satisfies the first-order equation \( \dot{z} = \ell[N - z - x_0 \exp(-kz/\ell)] \).

d) Show that this equation can be nondimensionalized to

\[
\frac{du}{d\tau} = a - bu - e^{-u}
\]

by an appropriate rescaling.

e) Show that \( a \geq 1 \) and \( b > 0 \).

f) Determine the number of fixed points \( u^* \) and classify their stability.

g) Show that the maximum of \( \dot{u}(t) \) occurs at the same time as the maximum of both \( \dot{z}(t) \) and \( y(t) \). (This time is called the peak of the epidemic, denoted \( t_{\text{peak}} \). At this time, there are more sick people and a higher daily death rate than at any other time.)

h) Show that if \( b < 1 \), then \( \dot{u}(t) \) is increasing at \( t = 0 \) and reaches its maximum at some time \( t_{\text{peak}} > 0 \). Thus things get worse before they get better. (The term epidemic is reserved for this case.) Show that \( \dot{u}(t) \) eventually decreases to 0.

