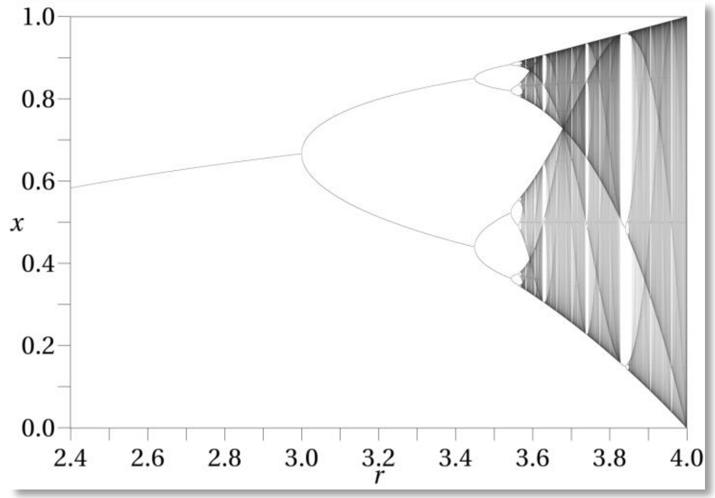

Nonlinear systems, chaos and control in Engineering

Module 1

One-dimensional nonlinear systems

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Schedule

Flows on the line

(Strogatz ch. 1 & 2)

(5 hs)

- Introduction
- Fixed points and linear stability
- Solving equations with computer
- Examples

Bifurcations

(Strogatz ch. 3)

(3 hs)

- Introduction
- Saddle-node
- Transcritical
- Pitchfork
- Examples

Flows on the circle

(Strogatz ch. 4)

(2 hs)

- Introduction to phase oscillators
- Nonlinear oscillator
- Fireflies and entrainment

Reminder Summary Part 1

- Flows on the line = first-order ordinary differential equations.

  $dx/dt = f(x)$

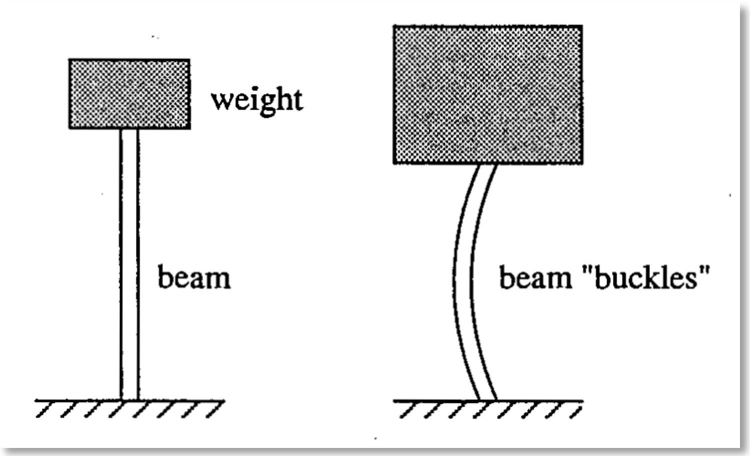
- Fixed point solutions: $f(x^*) = 0$
  - stable if $f'(x^*) < 0$
  - unstable if $f'(x^*) > 0$
  - neutral (bifurcation point) if $f'(x^*) = 0$
- There are no periodic solutions; the approach to fixed point solutions is monotonic (sigmoidal or exponential).
- In a 2D system a delayed control term can reduce oscillations; in a 1D system it can induce oscillations.

Outline

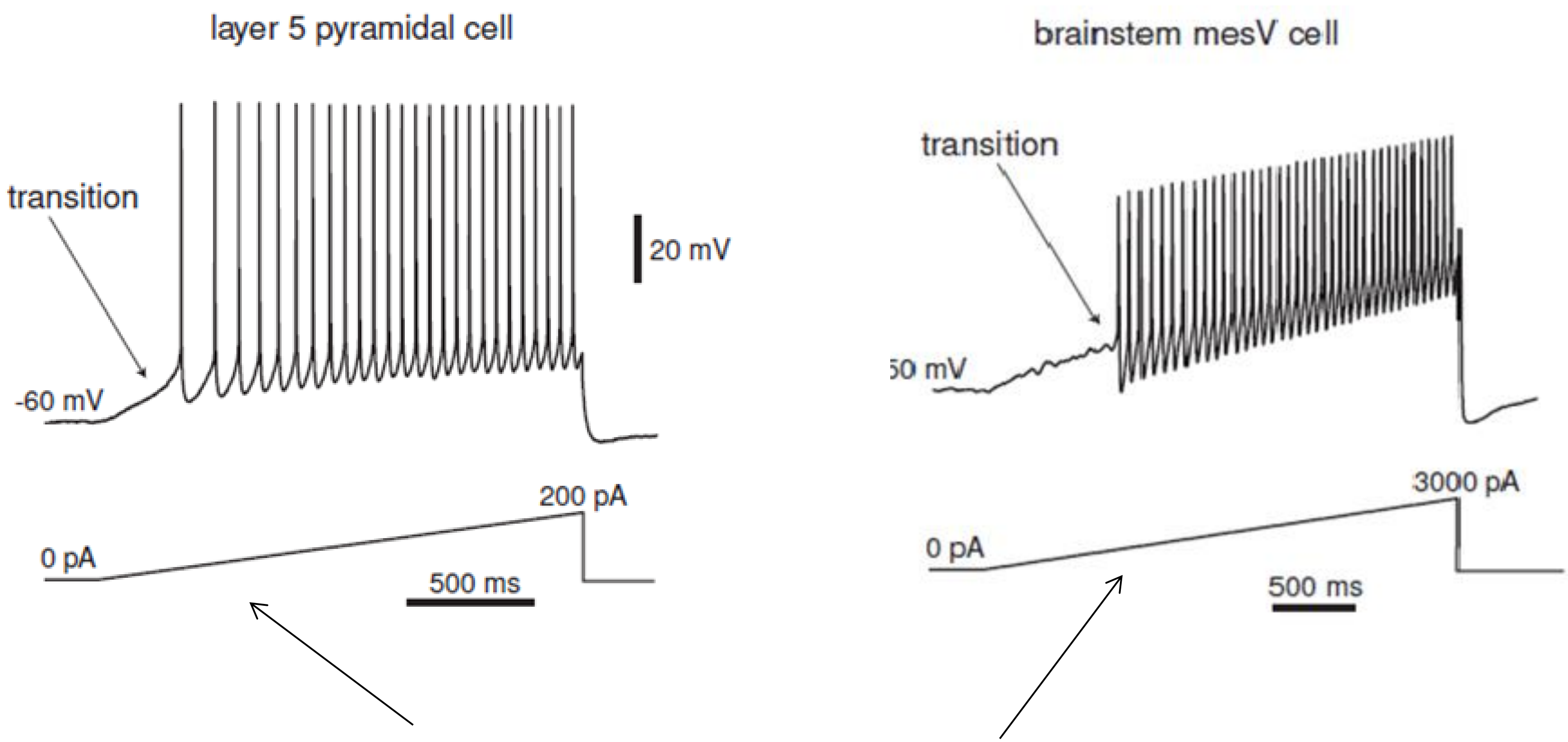
- Introduction to bifurcations
- Saddle-node, transcritical and pitchfork bifurcations
- Examples
  - Saddle-node: neuron model
  - Transcritical: laser threshold
  - Pitchfork: particle in a rotating wire hoop
- Imperfect bifurcations & catastrophes
  - Example: insect outbreak
- Delayed feedback examples

Bifurcations in 1D systems

- We consider a system with a control parameter.
- Bifurcation: a qualitative change (in the structure of the phase space) when the **control parameter is varied**:
  - Fixed points can be created or destroyed
  - The stability of a fixed point can change
- There are many examples in physical systems, biological systems, etc.

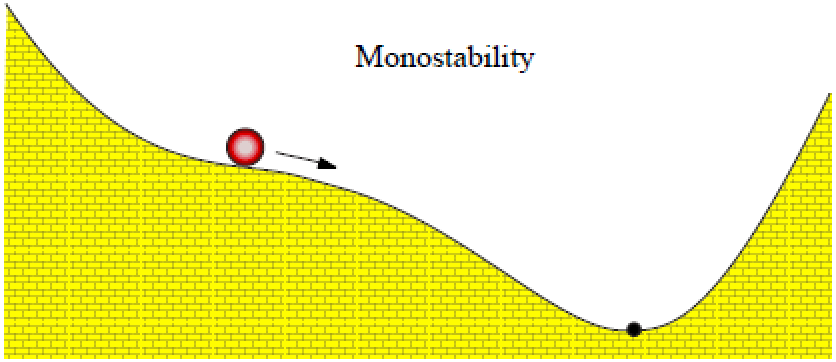


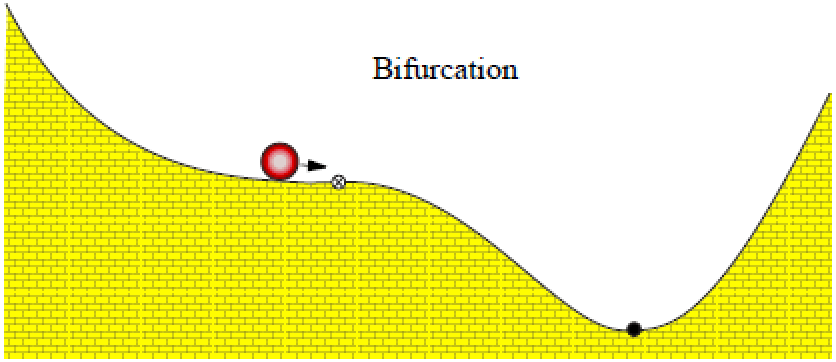
Example

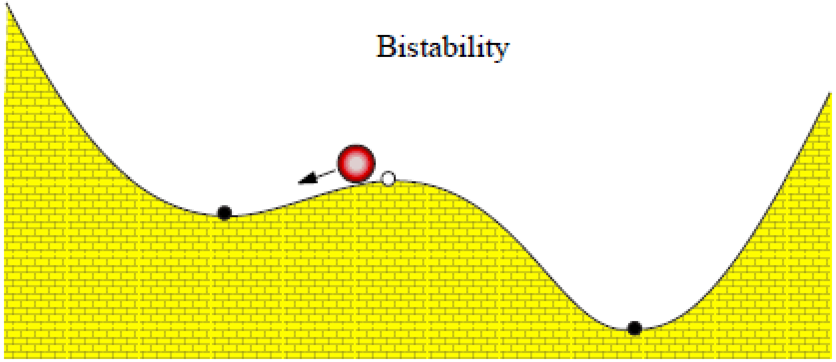


Control parameter increases in time

Bifurcation and potential

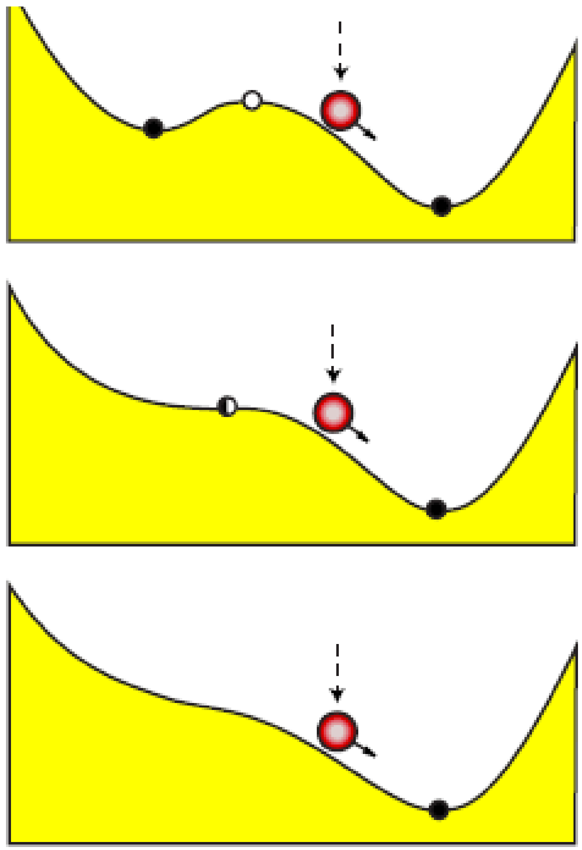




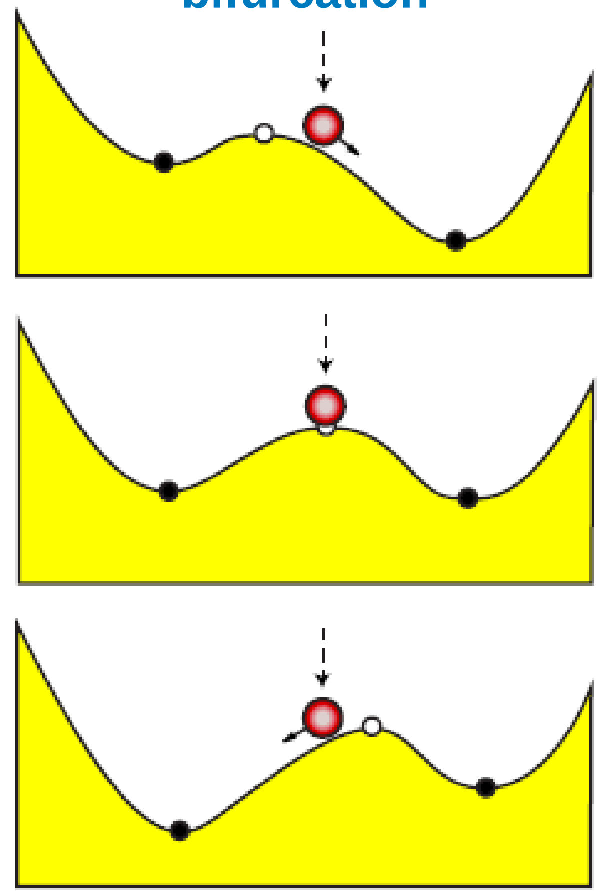


Bifurcations are not equivalent to qualitative change of behavior

Bifurcation but no change of behavior



Change of behavior but no bifurcation



Outline

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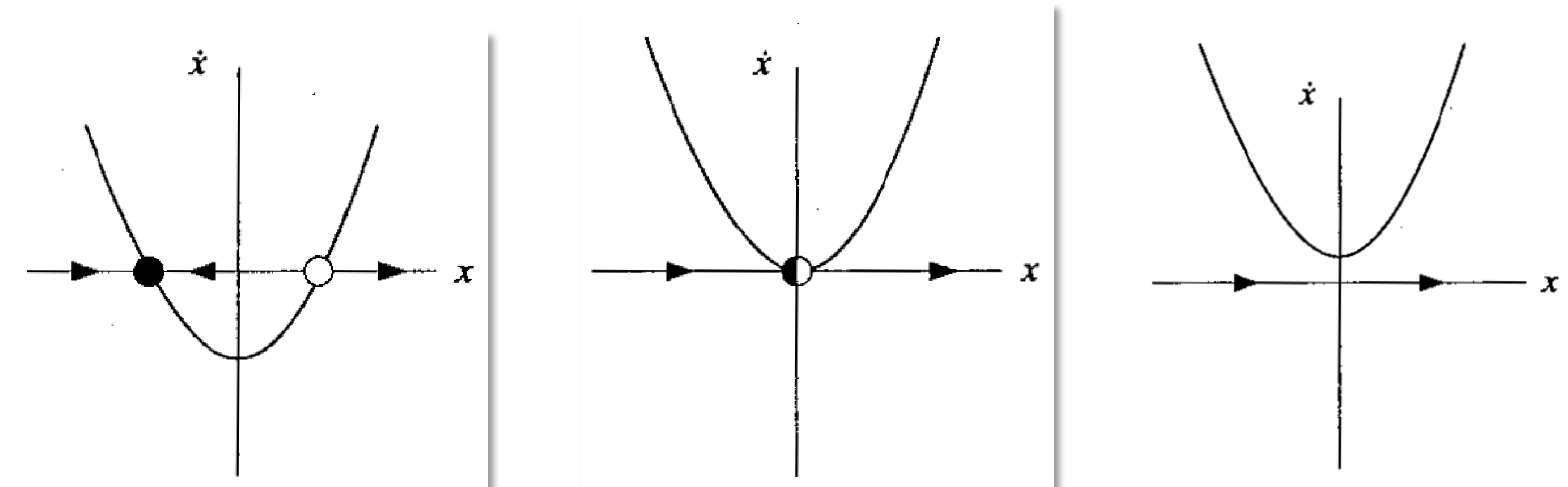
Saddle-node bifurcation

Basic mechanism for the creation or the destruction of fixed points

$$\dot{x} = f(x) = r + x^2 \qquad x^* = \pm\sqrt{-r}$$



(a) $r<0$ (b) $r=0$ (c) $r>0$

$$f'(-\sqrt{-r}) = -2\sqrt{-r}$$ Stable if r<0

$$f'(\sqrt{-r}) = 2\sqrt{-r}$$ unstable

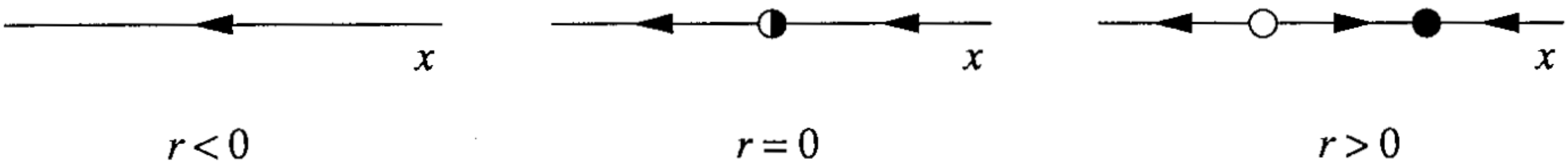
At the bifurcation point $r^*=0$: $f'(x^*) = 0$

Example

$$\dot{x} = r - x^2$$

- Calculate the fixed points and their stability as a function of the control parameter *r*

$$x^* = \pm\sqrt{r}$$



Normal forms

- Are representative of all saddle-node bifurcations.

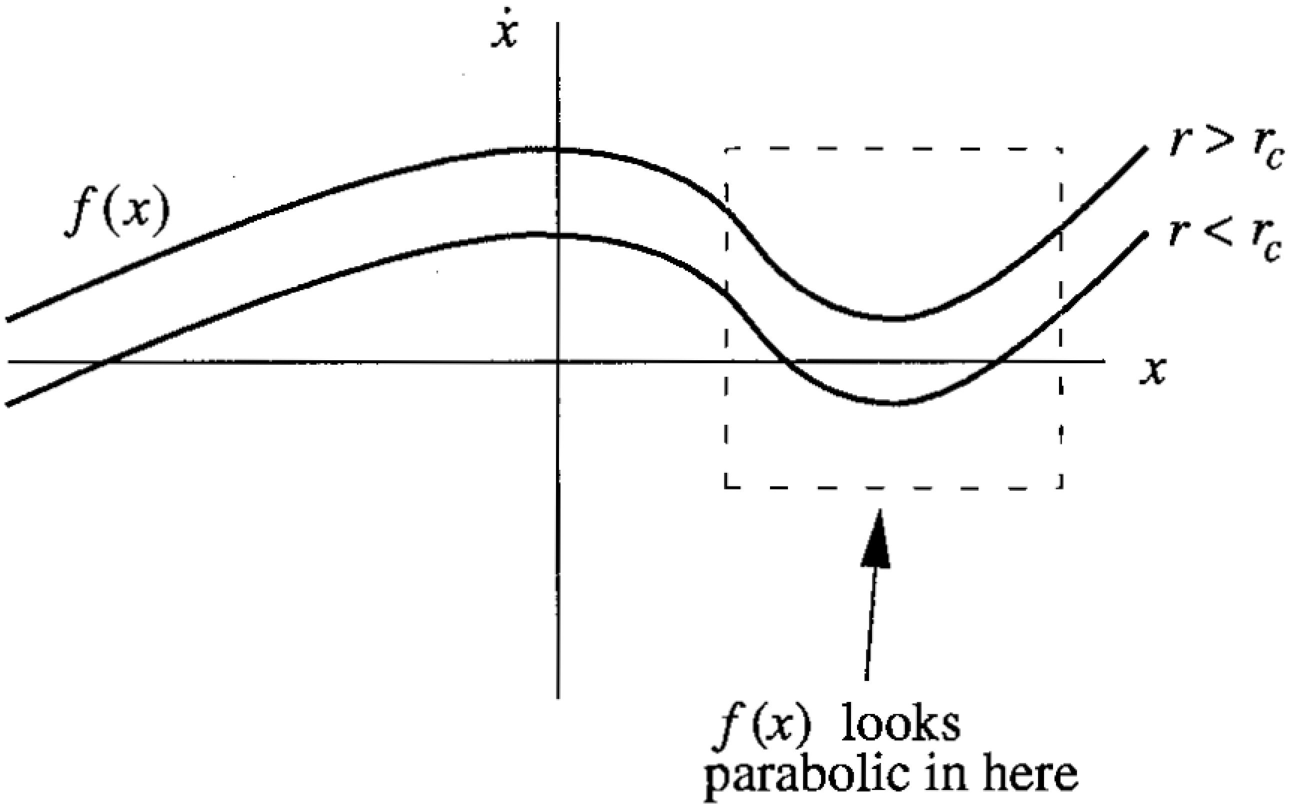
- Close to the saddle-node bifurcation the dynamics can be approximated by

$$\dot{x} = r - x^2$$ or $$\dot{x} = r + x^2$$

Example:

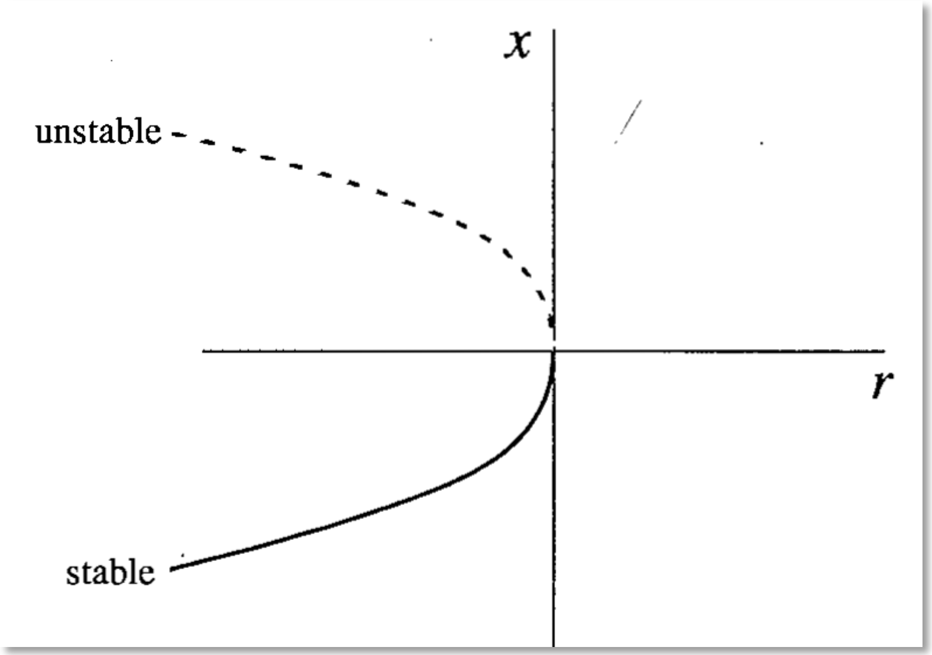
$$\dot{x} = r - x - e^{-x}$$
$$= r - x - \left[1 - x + \frac{x^2}{2!} + \cdots\right]$$
$$= (r - 1) - \frac{x^2}{2} + \cdots$$

Near a saddle-node bifurcation



Bifurcation diagram

$$\dot{x} = r + x^2$$



Two fixed points → one fixed point → 0 fixed point

A pair of fixed points appear (or disappear) out of the “clear blue sky” (“blue sky” bifurcation, Abraham and Shaw 1988).

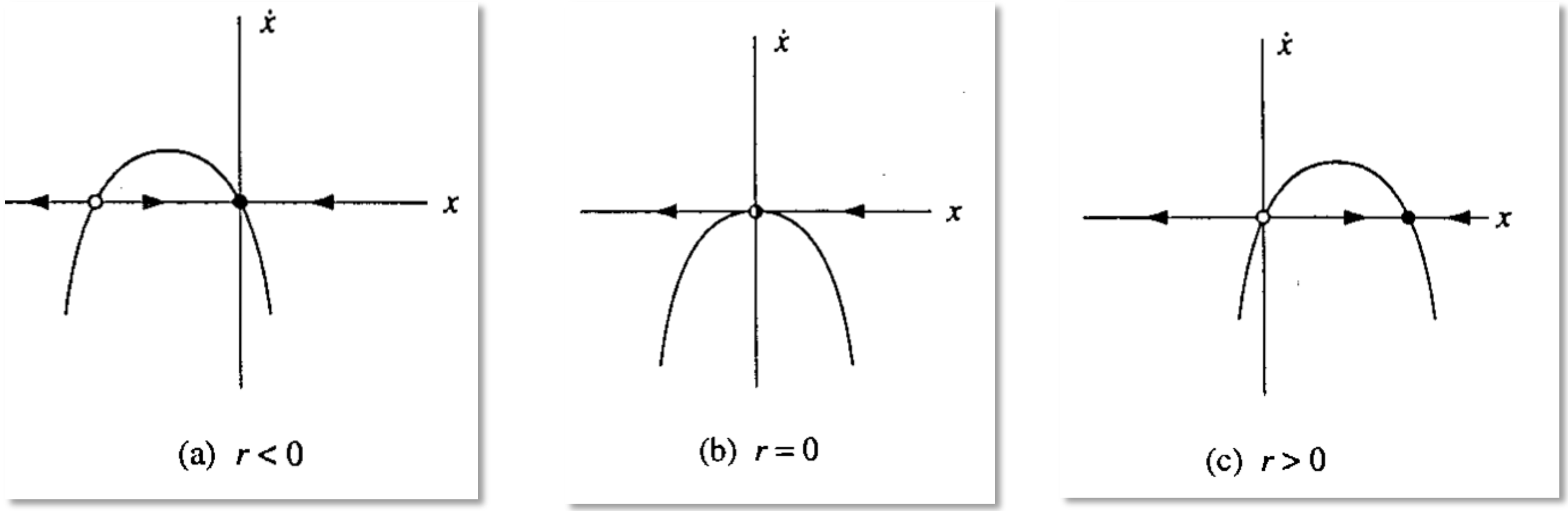
Transcritical bifurcation

$$\dot{x} = rx - x^2$$

$x^* = 0$

$x^* = r$

are the fixed points for all r



Transcritical bifurcation: general mechanism for changing the stability of fixed points.

Bifurcation diagram

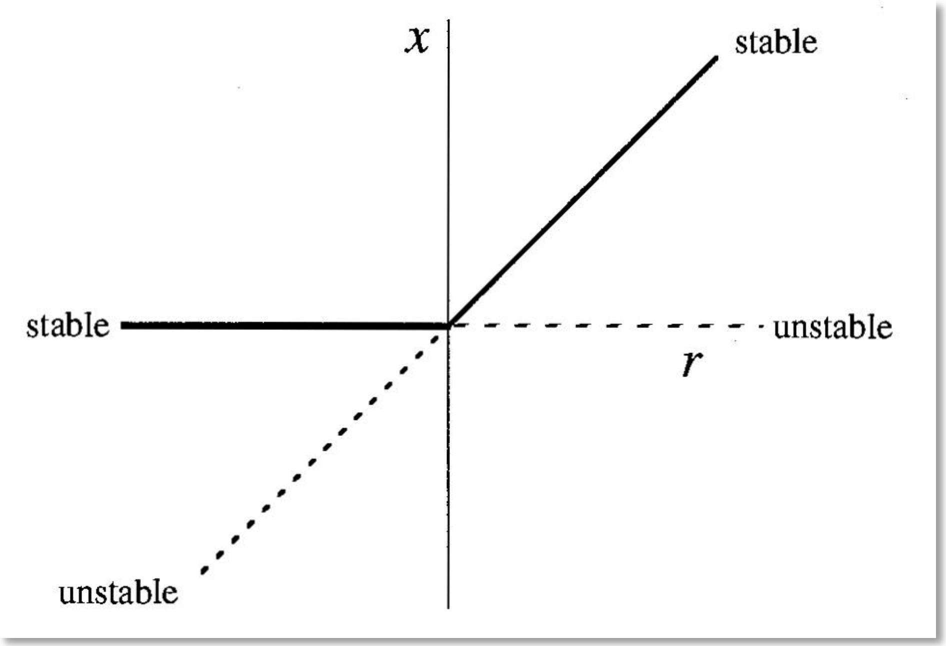
$$\dot{x} = rx - x^2$$

fixed points $x^* = 0$ and $x^* = r$

$$f'(x) = r - 2x$$

$$f'(0) = r$$

$$f'(r) = -r$$



- Exchange of stability at $r = 0$.
- **Exercise**: $\dot{x} = r \ln x + x - 1$

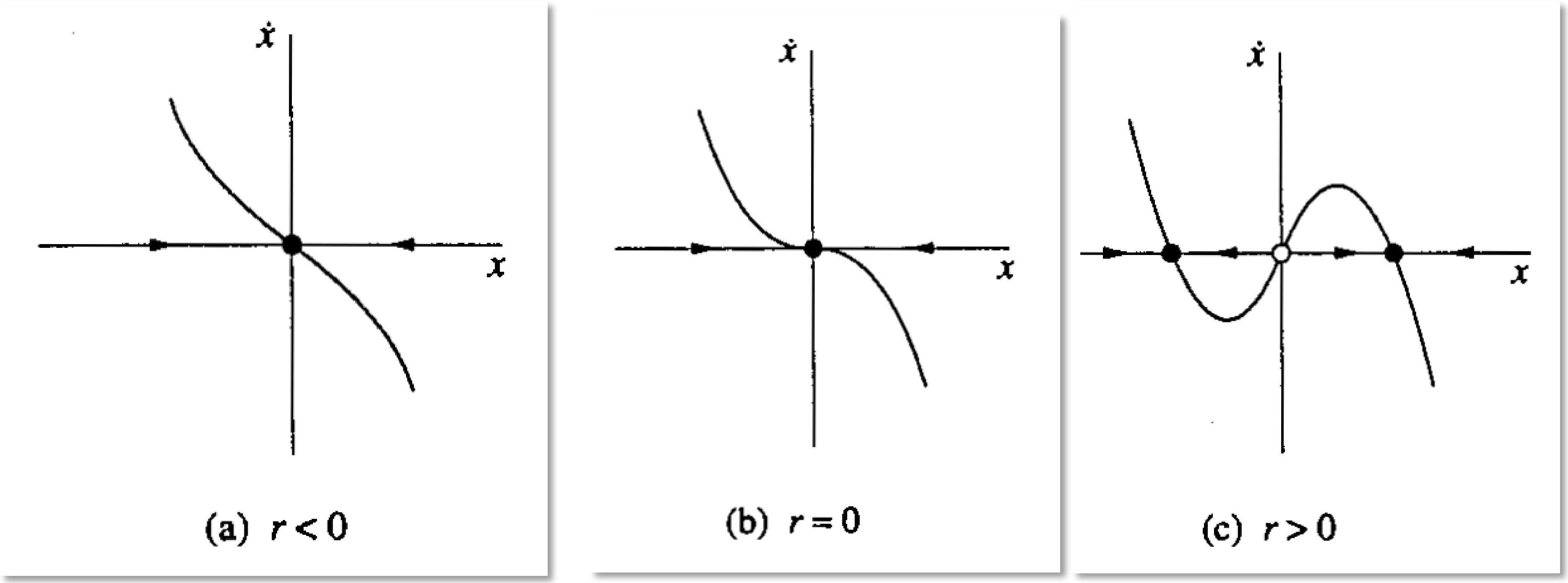
  show that a transcritical bifurcation occurs near x=1

  (hint: consider $u = x\text{-}1$ small)

Pitchfork bifurcation

$$\dot{x} = rx - x^3$$

Symmetry x → -x



One fixed point → 3 fixed points

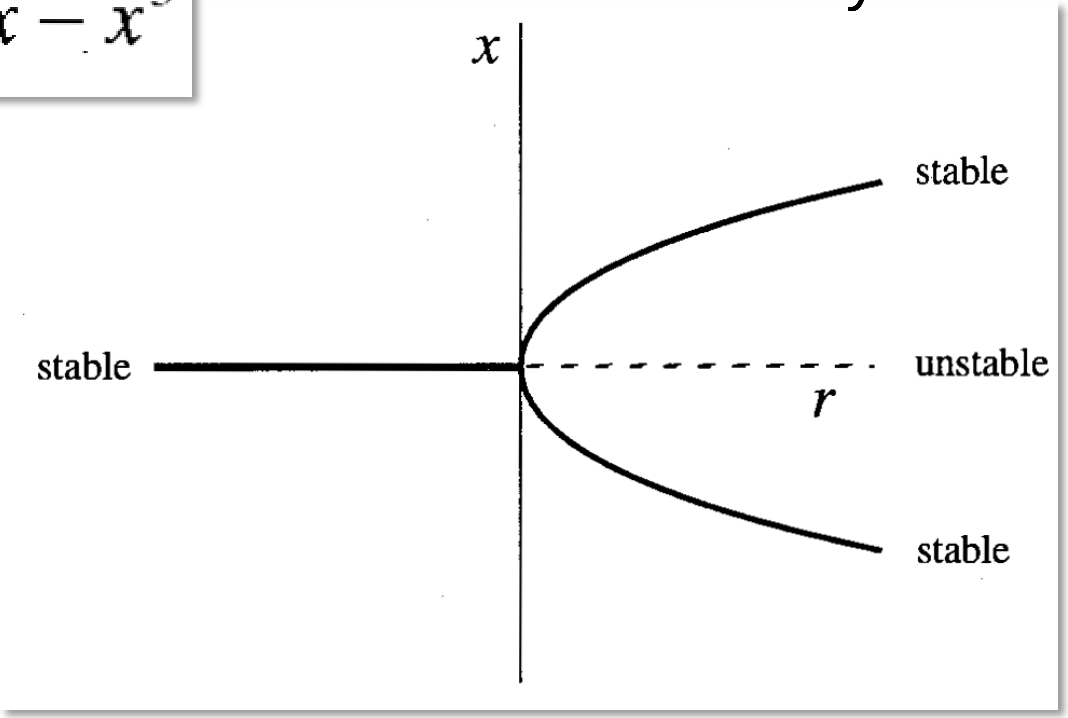
Bifurcation diagram

$$\dot{x} = rx - x^3$$

Bistability

$$x^* = 0$$

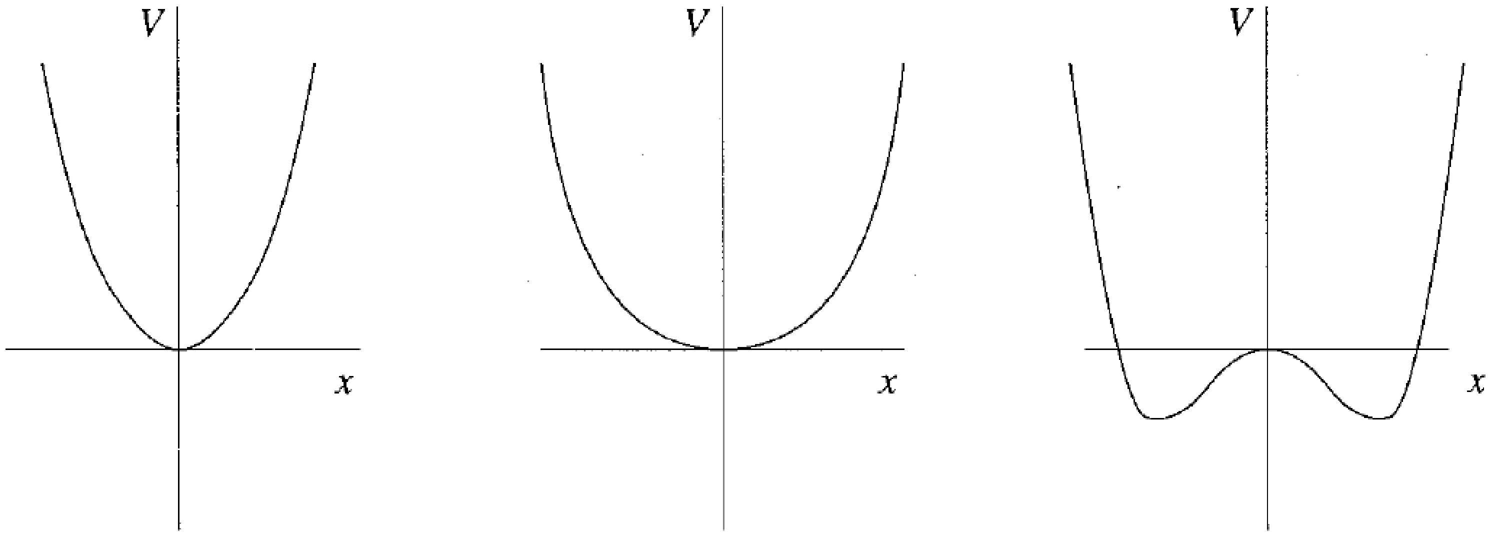
$$x^* = \pm\sqrt{-r}$$



- The governing equation is symmetric: *x* → -*x* but for *r* > 0: symmetry broken solutions.

Potential

$\dot{x} = rx - x^3$ $\qquad \dot{x} = f(x)$ $\qquad f(x) = -dV/dx$ $\qquad V(x) = -\frac{1}{2}rx^2 + \frac{1}{4}x^4$

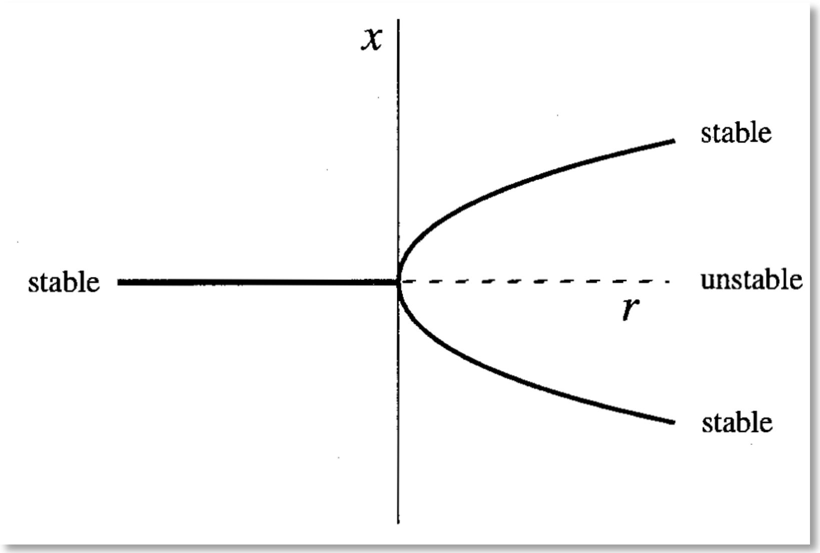


$r < 0$ $\qquad r = 0$ $\qquad r > 0$

Pitchfork bifurcations

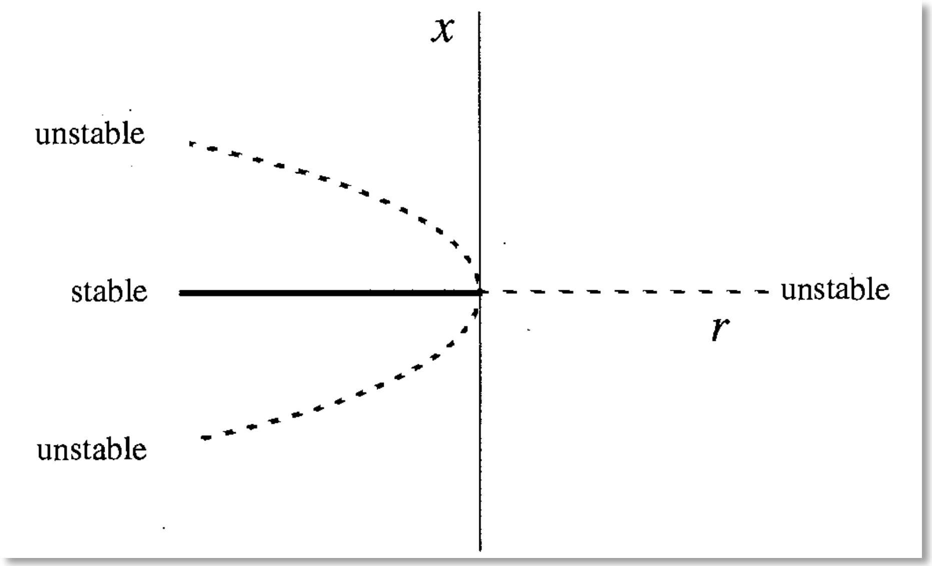
Supercritical:
x^3 is stabilizing

$$\dot{x} = rx - x^3$$



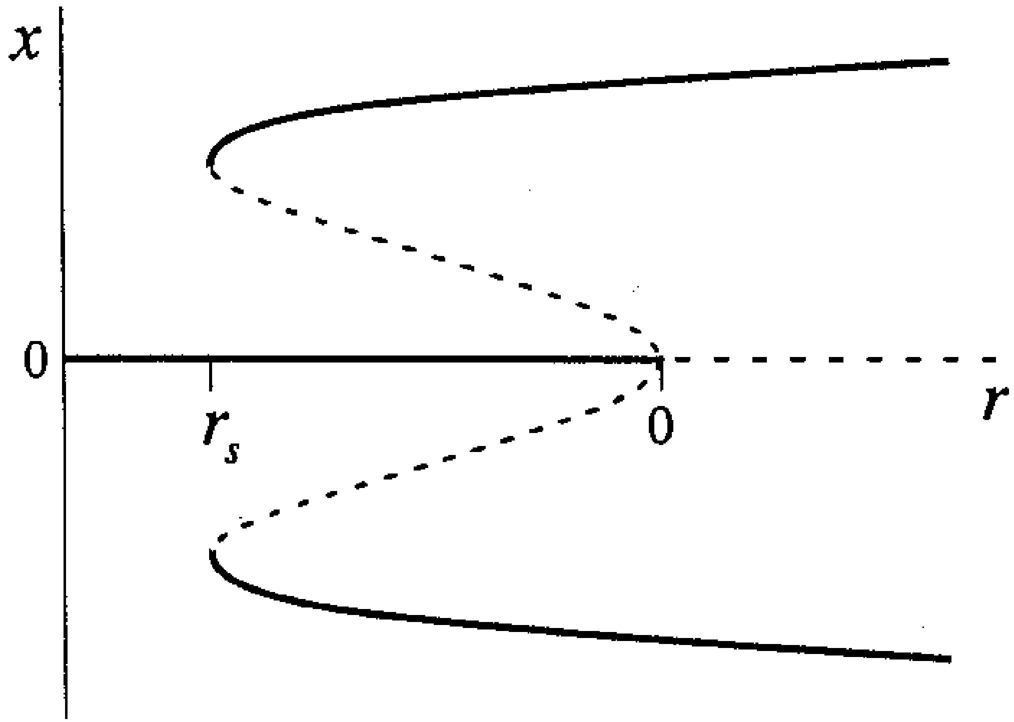
Subcritical:
x^3 is destabilizing

$$\dot{x} = rx + x^3$$

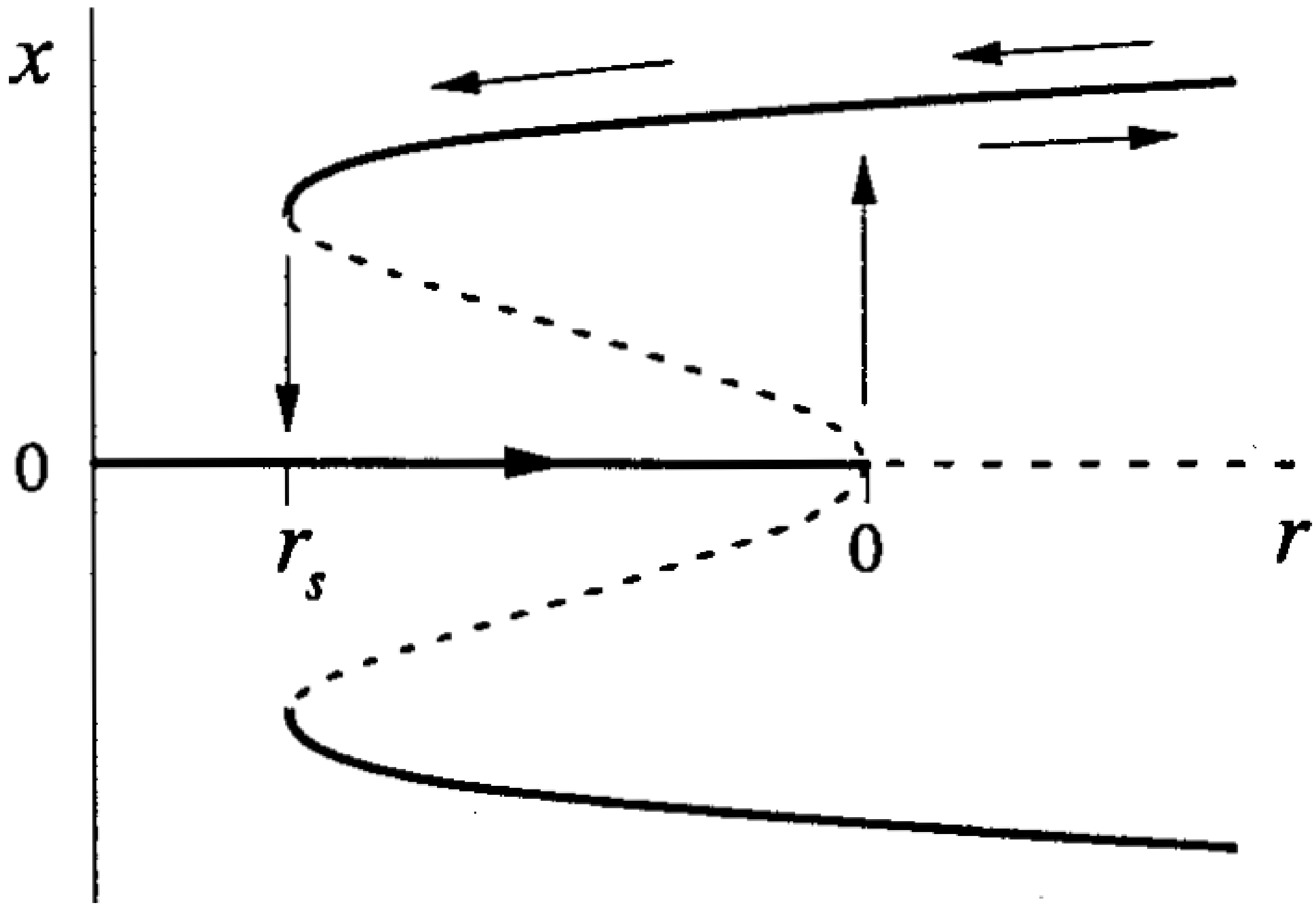


Exercise: find the fixed points and compute their stability

$$\dot{x} = rx + x^3 - x^5$$

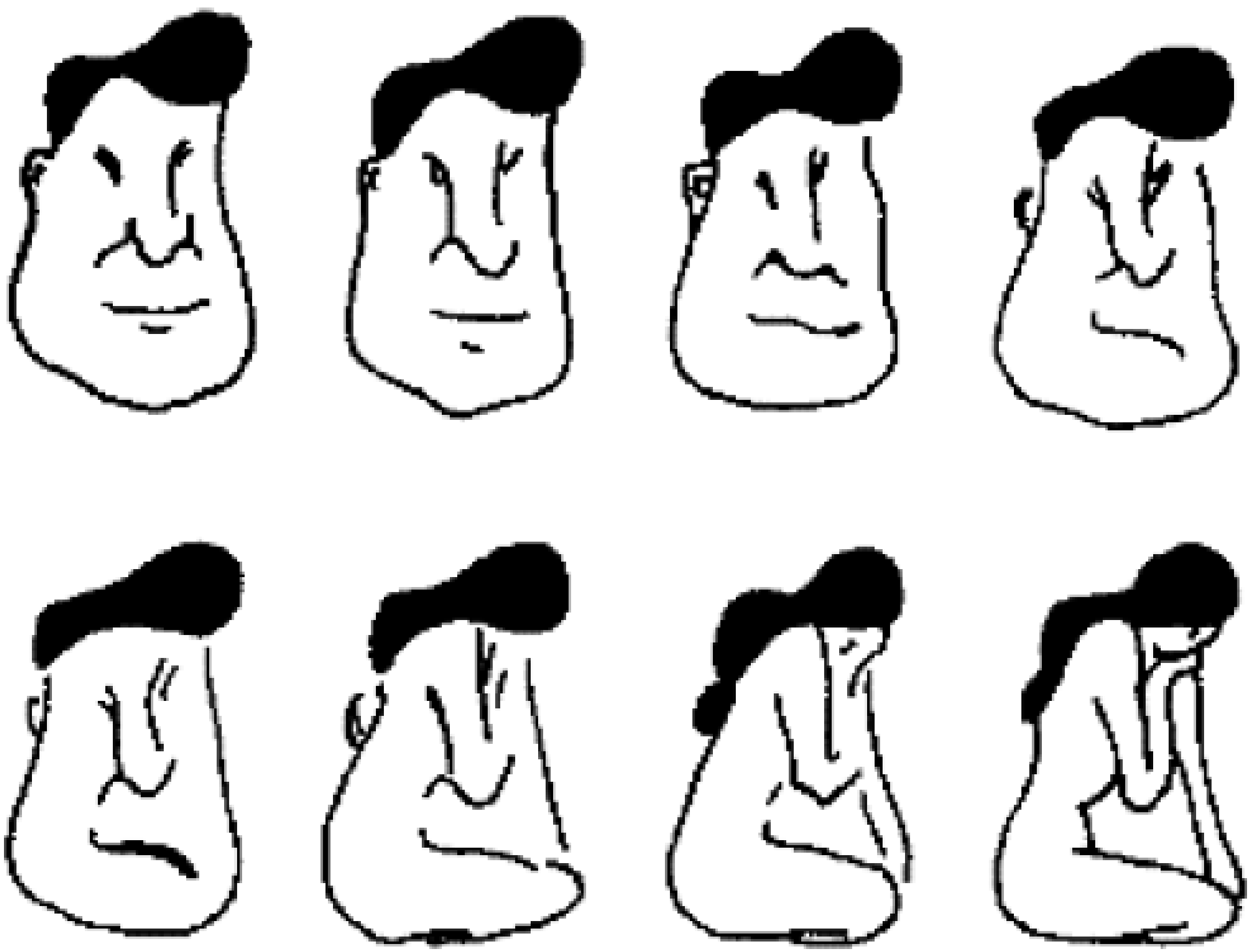


Subcritical bifurcation: Hysteresis



Critical or dangerous transition! A lot of effort in trying to find “early warning signals” (more latter)

Hysteresis: sudden changes in visual perception



Fischer (1967): experiment with 57 students.

"When do you notice an abrupt change in perception?"

Summary

- Bifurcation condition: change in the stability of a fixed point

$$f'(x^*) = 0$$

- In first-order ODEs: three possible bifurcations
  - Saddle node
  - Pitchfork
  - Trans-critical

- The normal form describes the behavior near the bifurcation.

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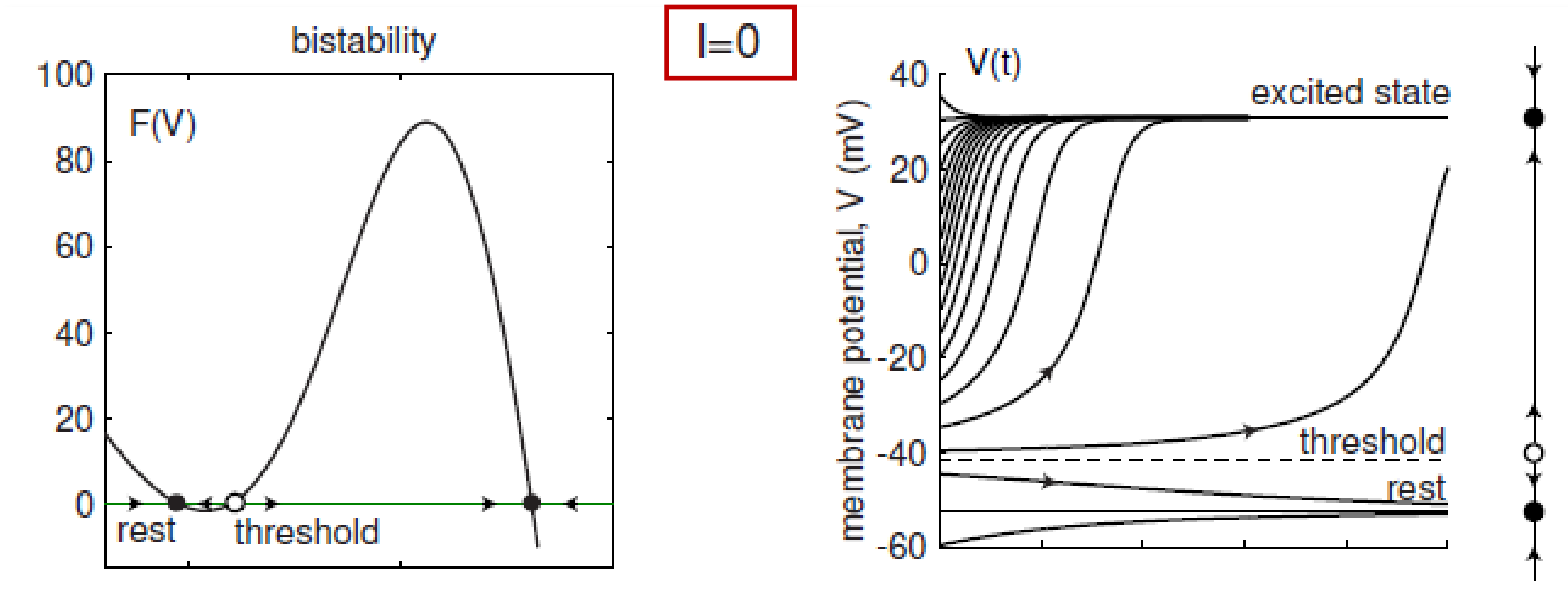
Example: neuron model

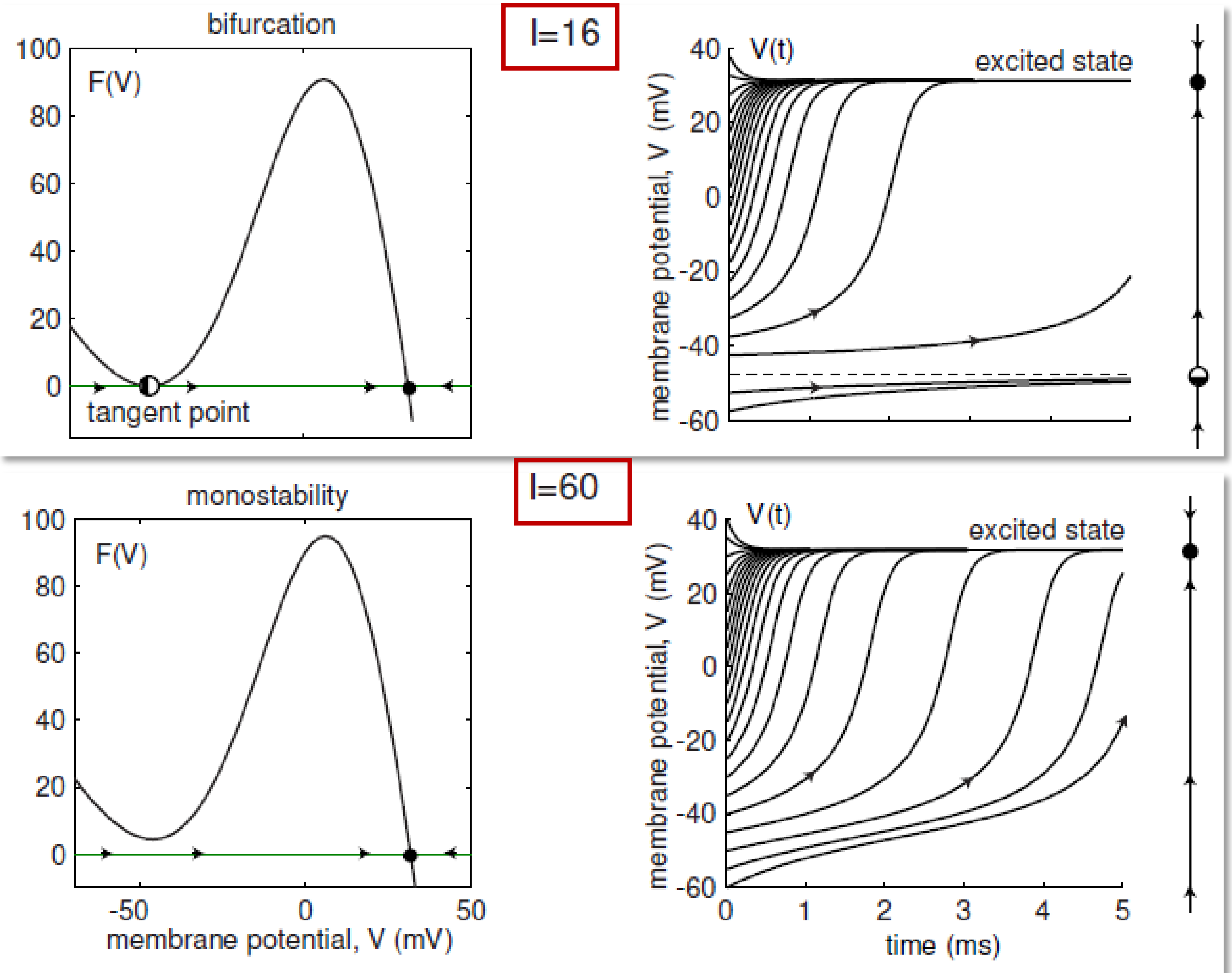
$$C\,\dot{V} = I - g_L(V - E_L) - \overbrace{g_{Na}\,m_\infty(V)\,(V - E_{Na})}^{\text{instantaneous } I_{Na,p}}$$

$$m_\infty(V) = 1/(1 + \exp\{(V_{1/2} - V)/k\})$$

$C = 10\ \mu F$, $I = 0$ pA, $g_L = 19$ mS, $E_L = -67$ mV,
$g_{Na} = 74$ mS, $V_{1/2} = 1.5$ mV, $k = 16$ mV, $E_{Na} = 60$ mV

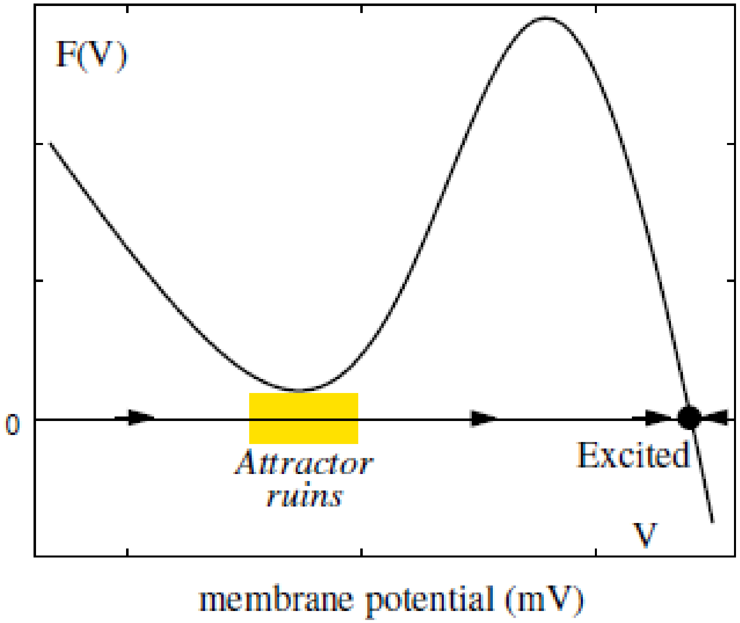


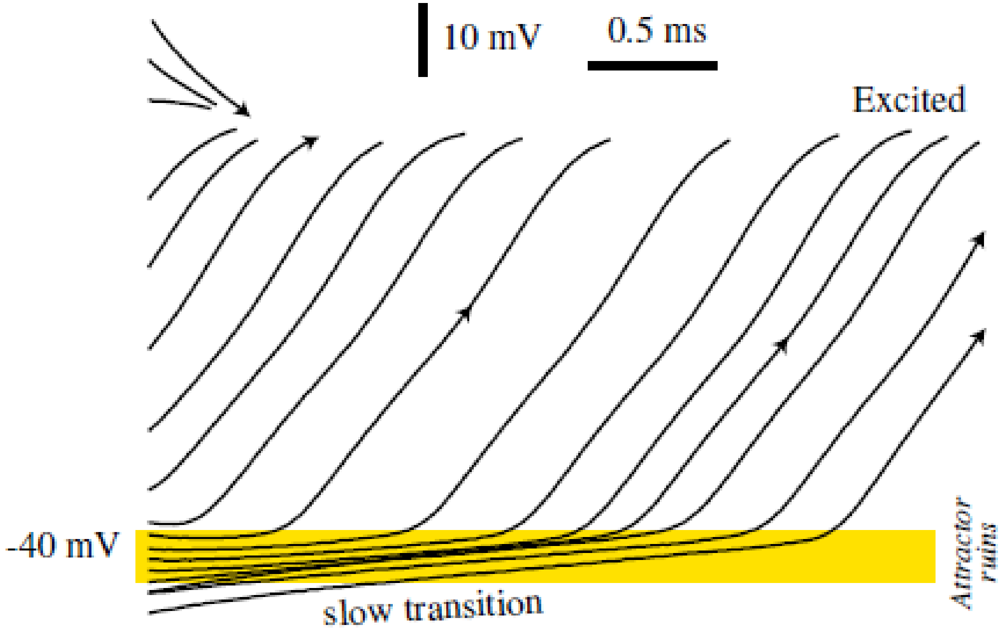
Bifurcation



Near the bifurcation point: slow dynamics

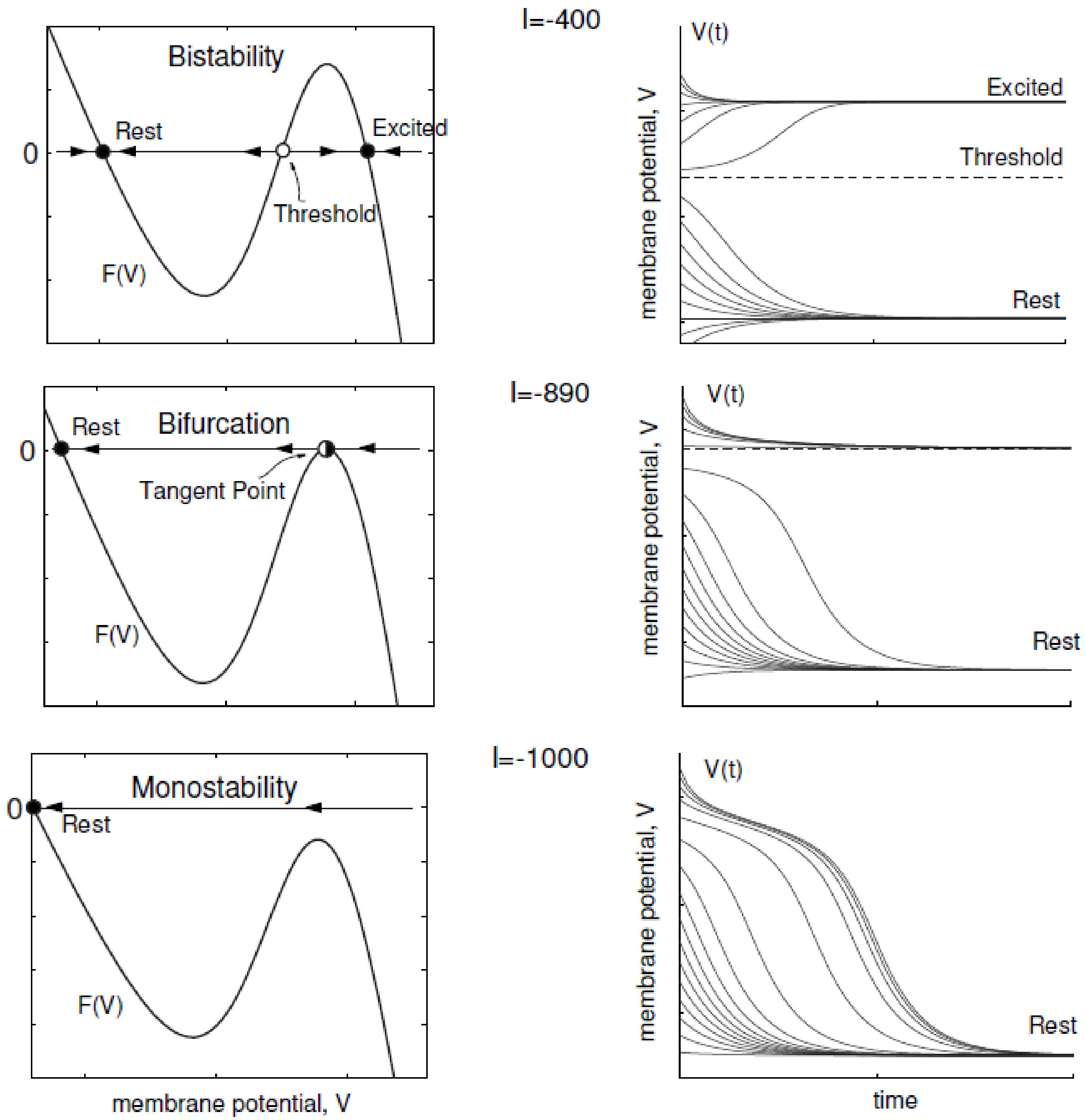
$I = 30 \text{ pA}$





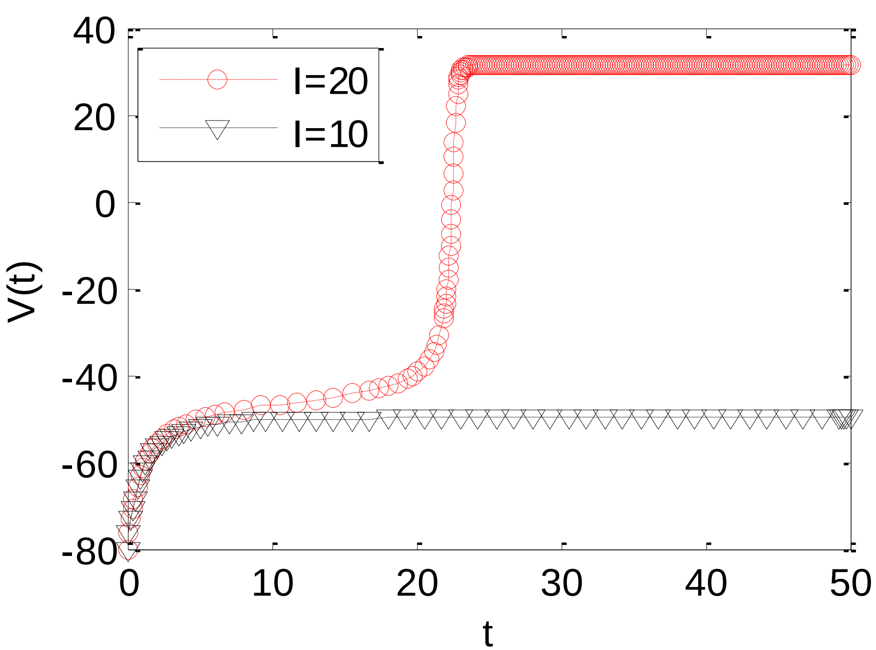
This slow transition is an "early warning signal" of a critical or dangerous transition ahead (more latter)

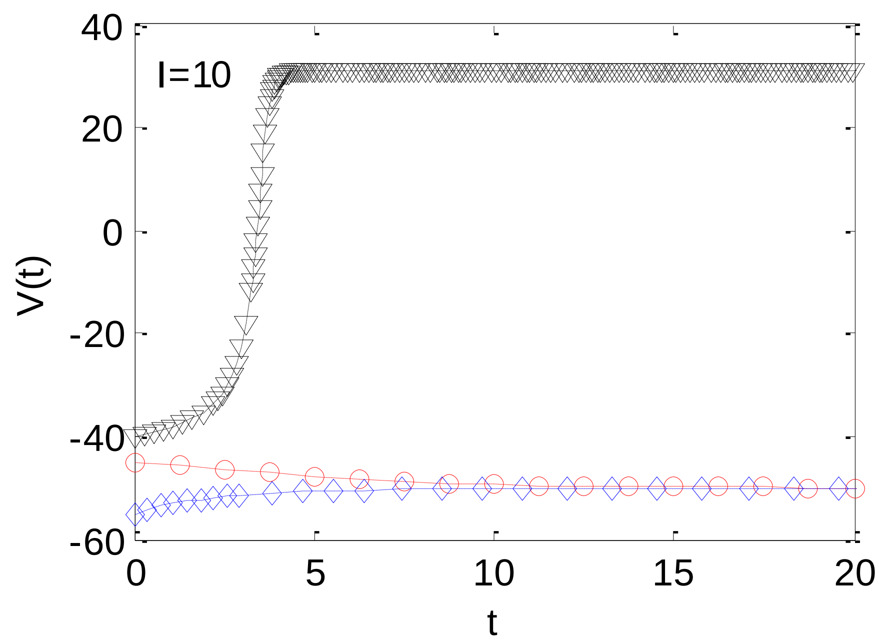
If the control parameter now decreases



Class/home work with matlab

- Simulate the neuron model with different values of the control parameter I and/or different initial conditions.





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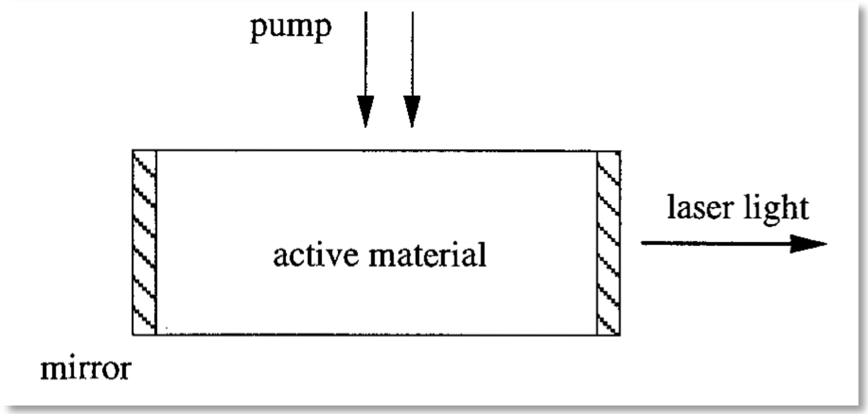
Example: laser threshold

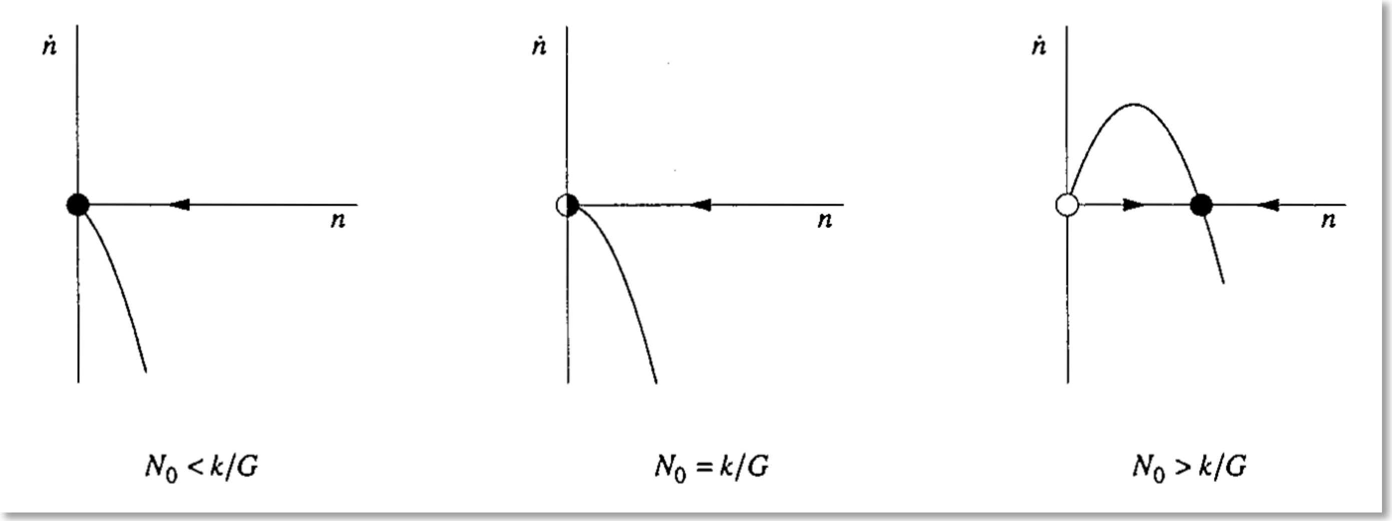
$$\dot{n} = \text{gain} - \text{loss}$$
$$= GnN - kn.$$



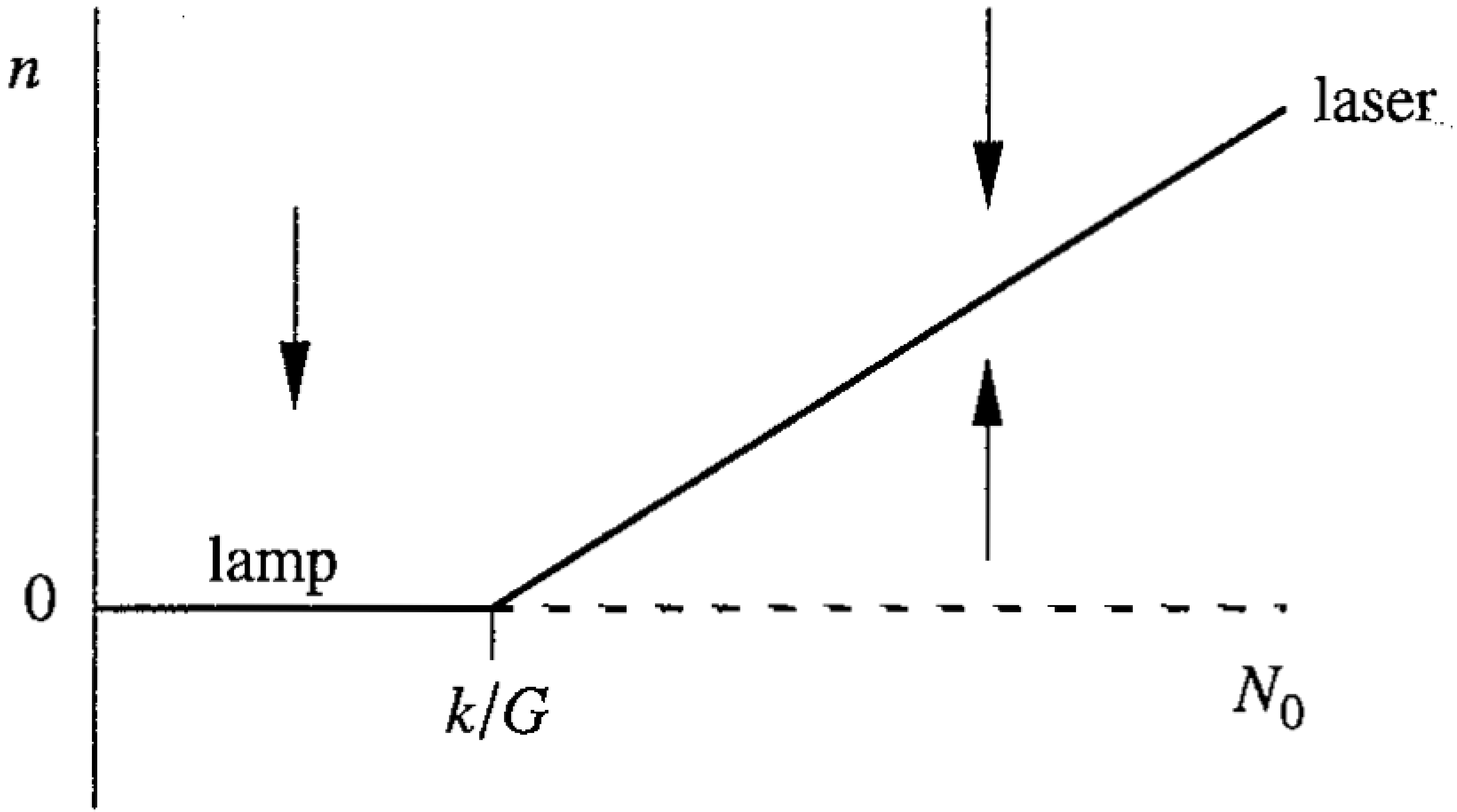
$$N(t) = N_0 - \alpha n$$

$$\dot{n} = Gn(N_0 - \alpha n) - kn$$
$$= (GN_0 - k)n - (\alpha G)n^2$$

$$\dot{x} = rx - x^2$$



Bifurcation diagram: LI curve



Laser turn-on delay

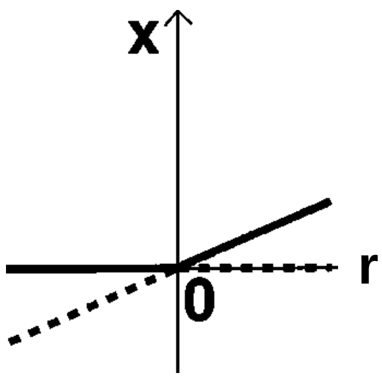
$$\dot{x} = rx - x^2$$

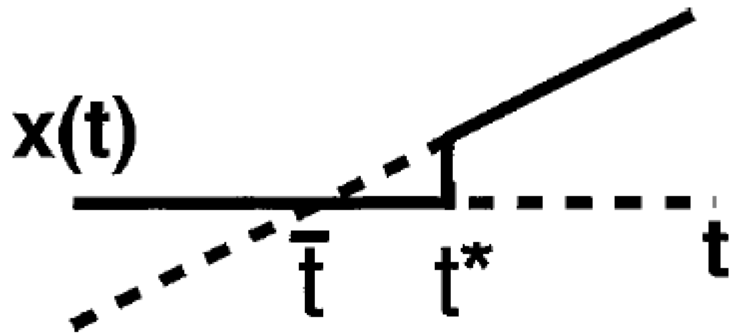
$$r(t) = r_0 + vt$$

Linear increase of control parameter

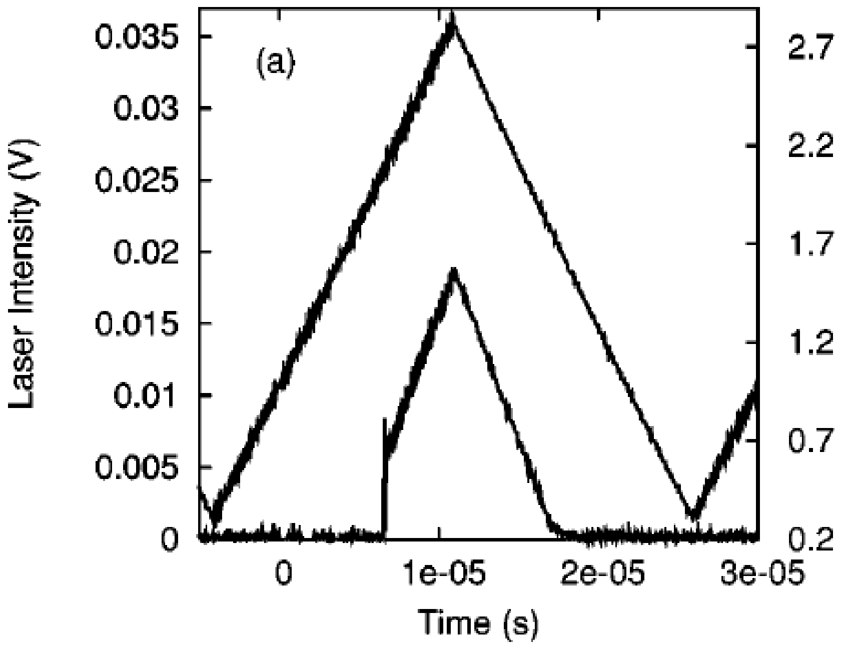
$$r_0 < r^* = 0$$

Start before the bifurcation point

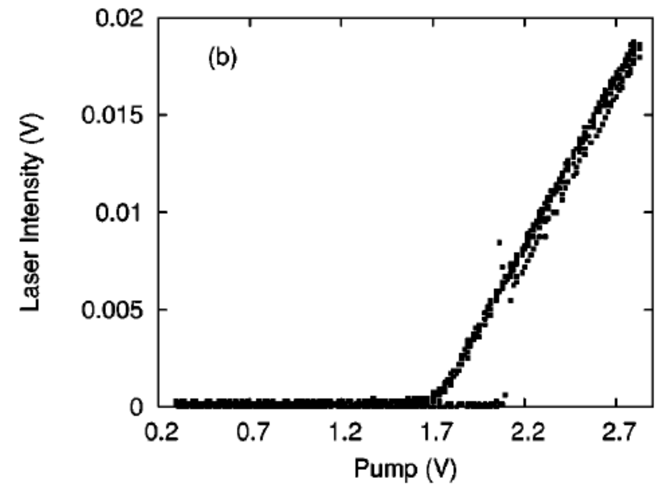




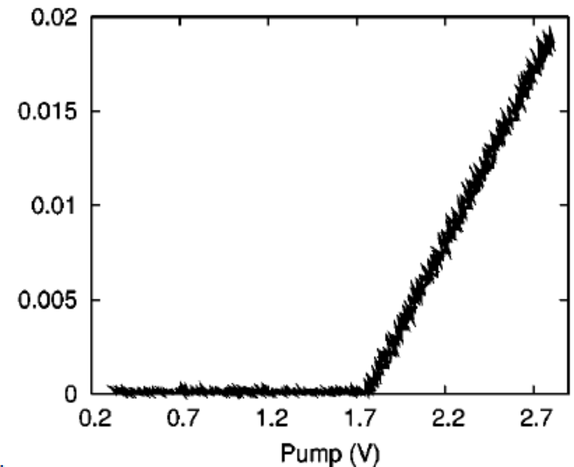
Comparison with experimental observations



Dynamical hysteresis



Quasi-static very slow variation of the control parameter



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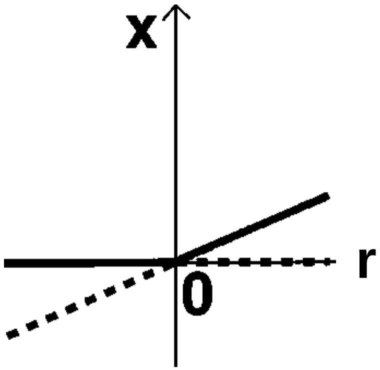
Laser turn-on



$$r(t) = r_0 \qquad r_0 > r^* = 0$$

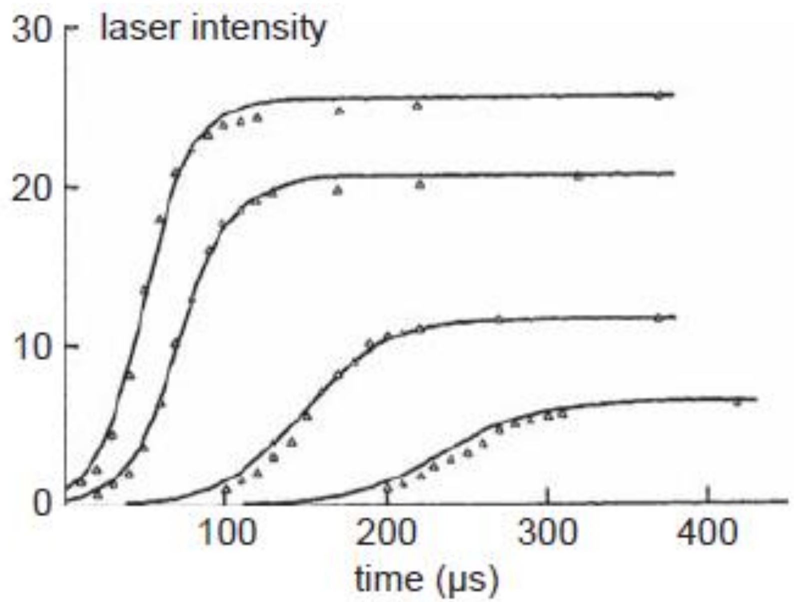
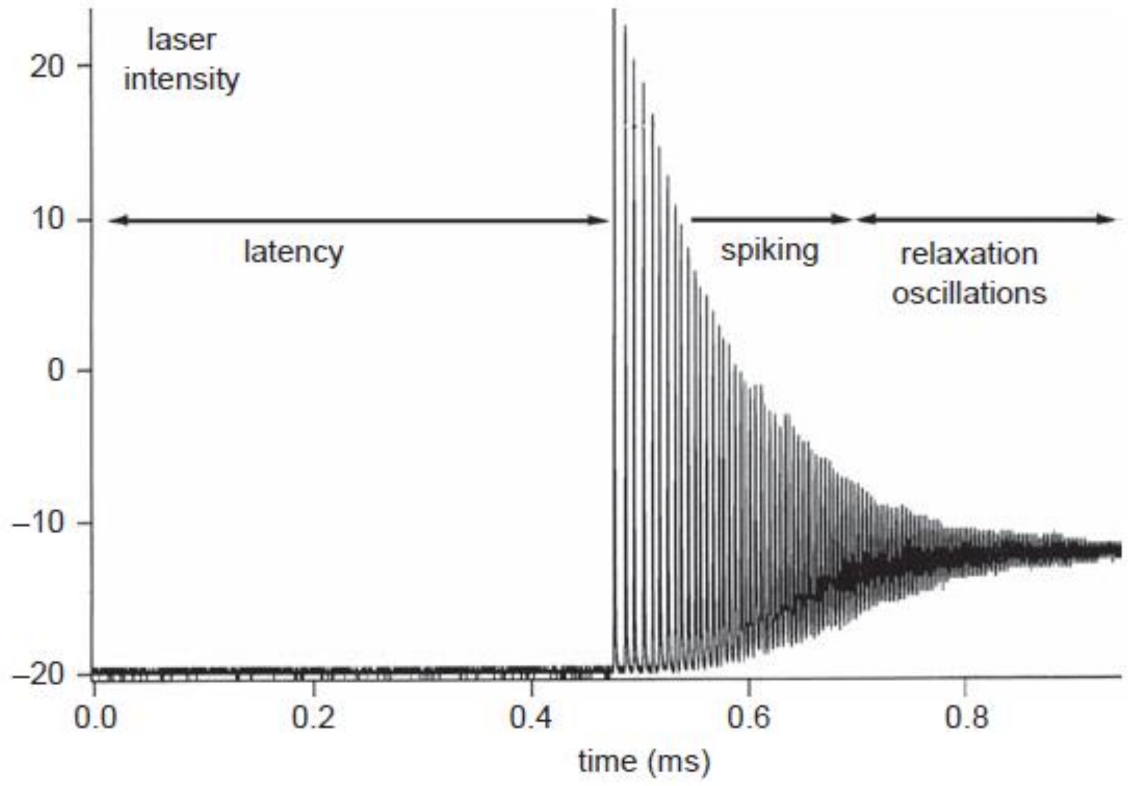


Fig. 1.3 He-Ne gas laser output as a function of time. From the lower to the upper time traces, the pump parameter above threshold is gradually increased. Reprinted Figure 2 with permission from Pariser and Marshall [30]. Copyright 1965 by the American Institute of Physics.

Turn on transient of a diode laser: delay and relaxation oscillations

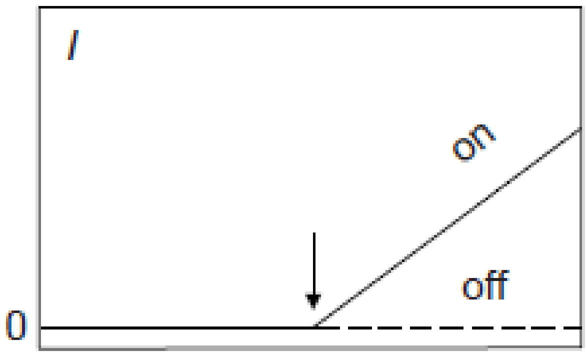


- We need more equations to explain these oscillations. The diode laser is not a 1D system.

With "noise": "imperfect" bifurcation

$$\dot{x} = f(x) + \xi(t)$$

$$\xi = 0$$



$$\xi \neq 0$$

$$\dot{n} = \text{gain} - \text{loss} + \beta$$

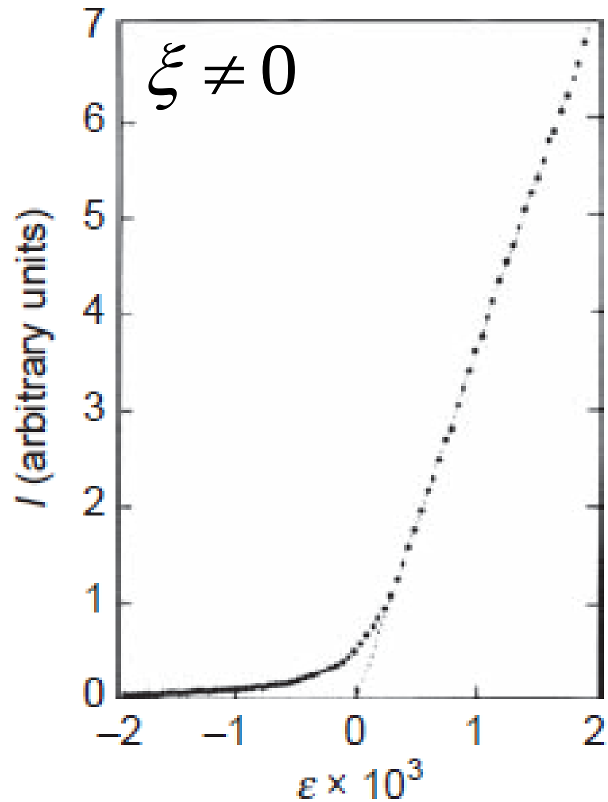


Fig. 1.17 Imperfect bifurcation for a laser in the presence of spontaneous emission, measured for a He-Ne laser. Reprinted Figure 1 with permission from Corti and Degiorgio [42]. Copyright 1976 by the American Physical Society.

Class/home work with matlab

$$\dot{x} = rx - x^2$$

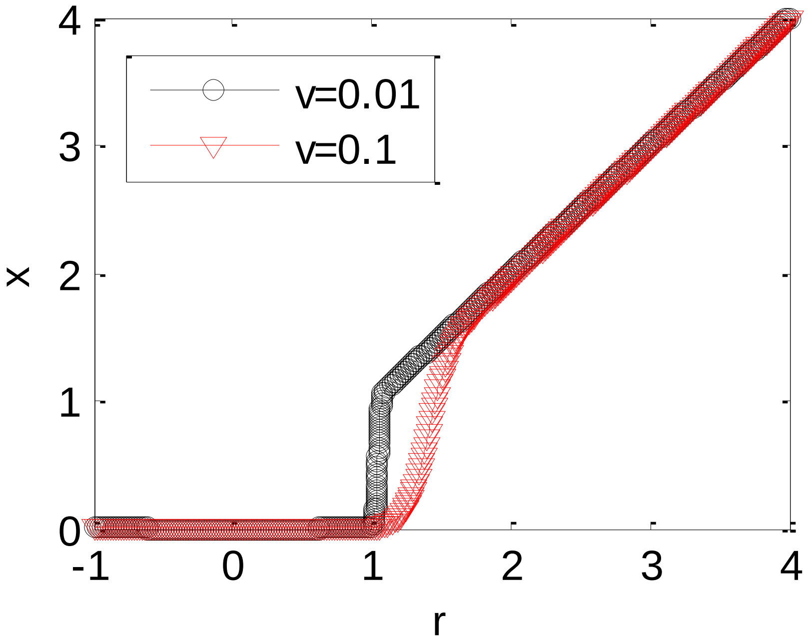
- Simulate the laser model when the control parameter *r* increases linearly in time. Consider different variation rate (*v*) and/or different initial value of the parameter (r_0).

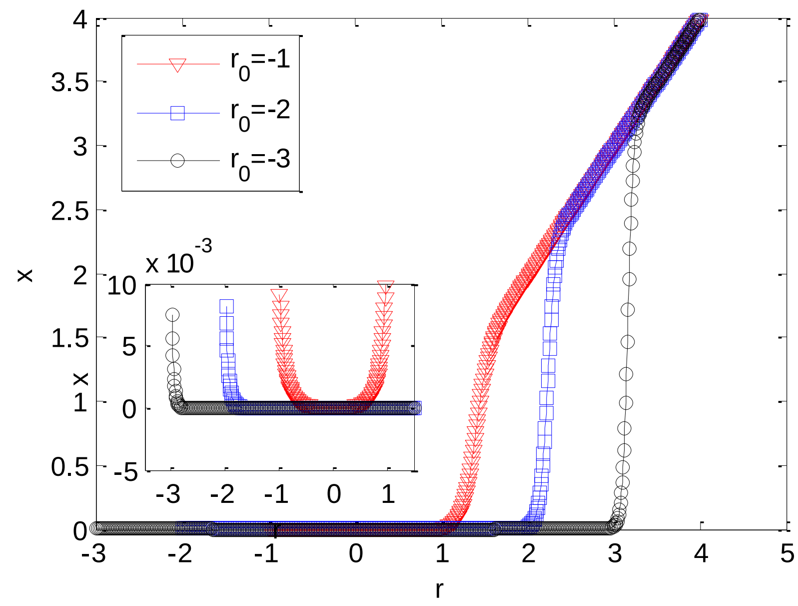
$$r(t) = r_0 + vt$$

$$x_0 = 0.01$$

$$v = 0.1$$

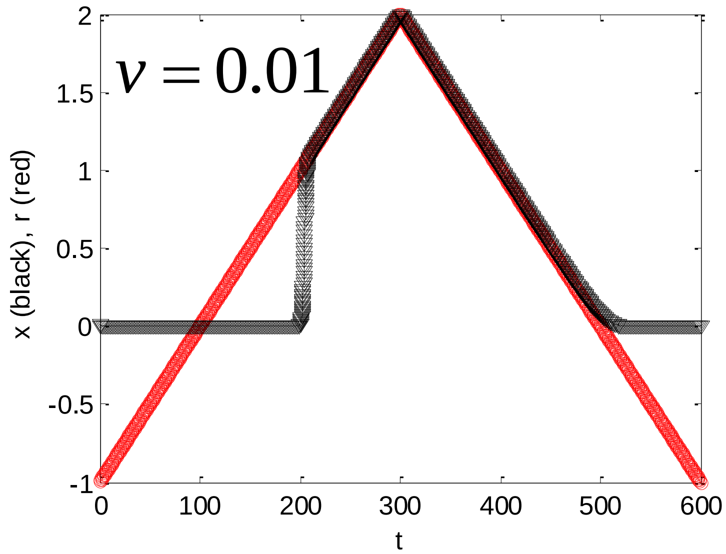
$$x_0 = 0.01$$

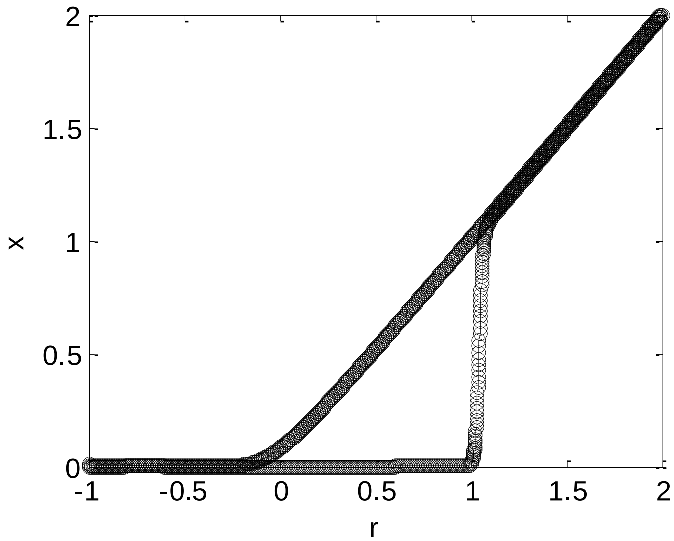




- Now consider that the control parameter *r* **increases and then decreases** linearly in time.

  Plot *x* and *r* vs time and plot *x* vs *r*.





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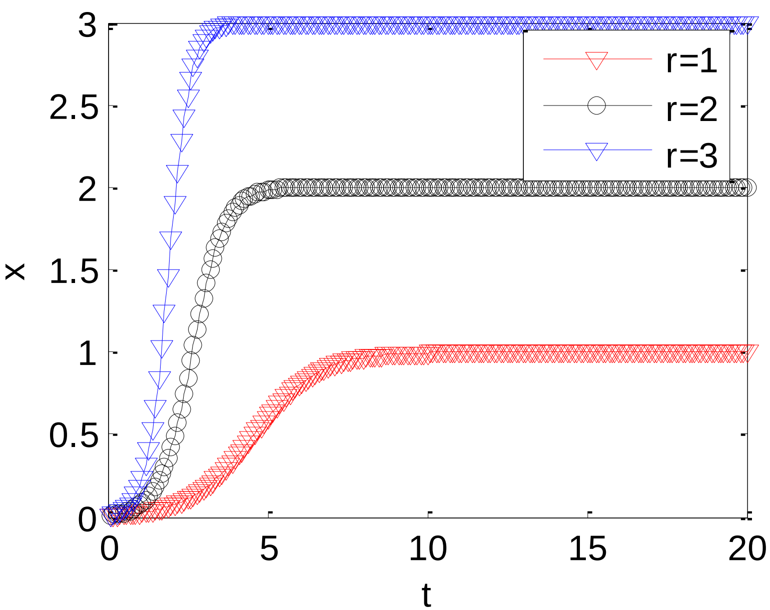
- Calculate the "turn on" when *r* is constant, *r*>*r**=0.

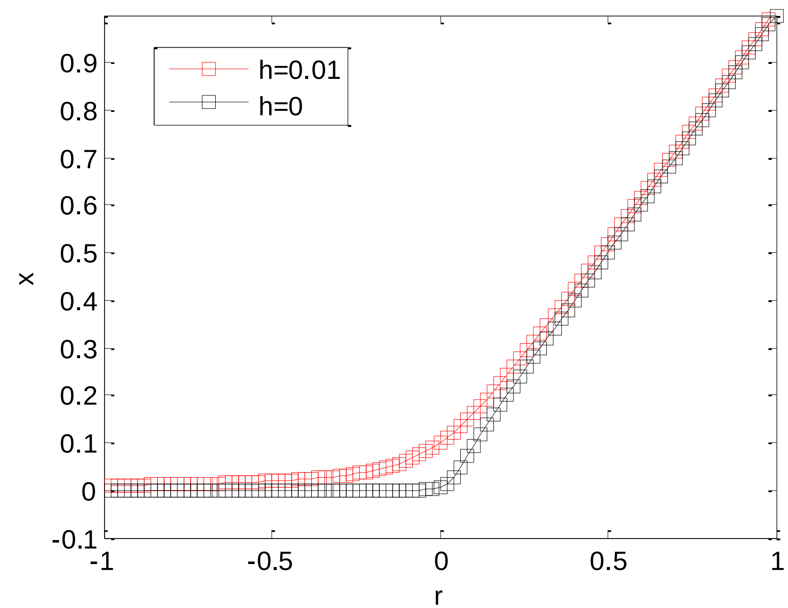
$$r(t) = r$$

$$x_0 = 0.01$$



- Calculate the bifurcation diagram by plotting *x(t=50)* vs *r*.

$$\dot{x} = rx - x^2 + h$$



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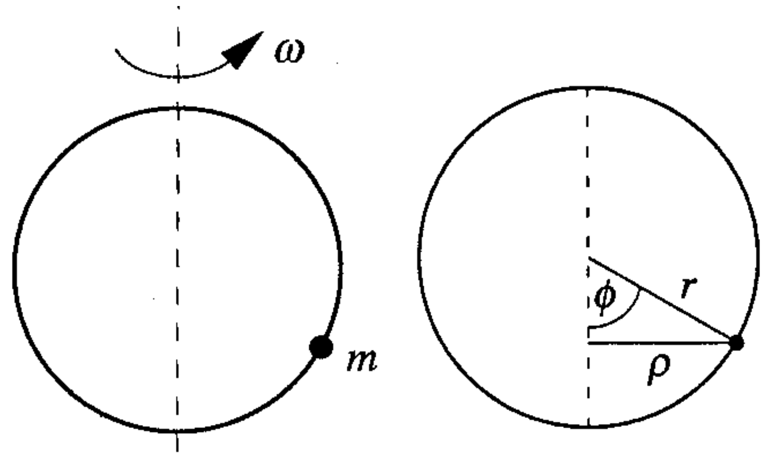
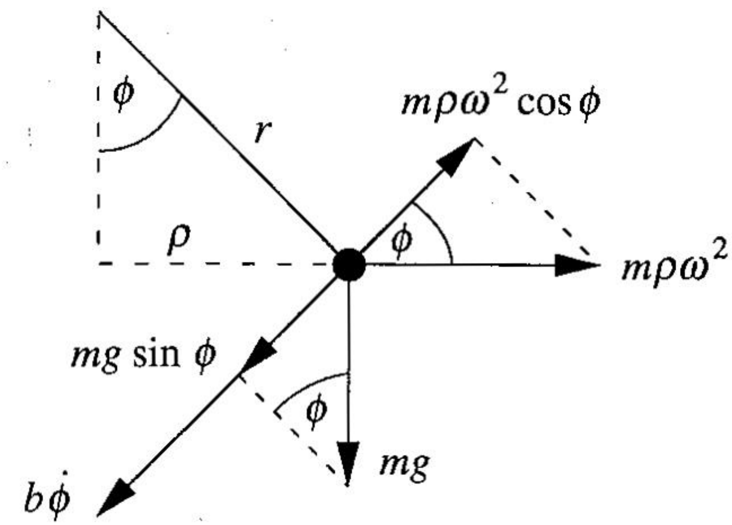
Example: particle in a rotating wire hoop

- A particle moves along a wire hoop that rotates at constant angular velocity





$$mr\ddot{\phi} = -b\dot{\phi} - mg\sin\phi + mr\omega^2 \sin\phi\cos\phi$$

- Neglect the second derivative (more latter)

$$b\dot{\phi} = -mg\sin\phi + mr\omega^2 \sin\phi\cos\phi$$

$$= mg\sin\phi\left(\frac{r\omega^2}{g}\cos\phi - 1\right)$$

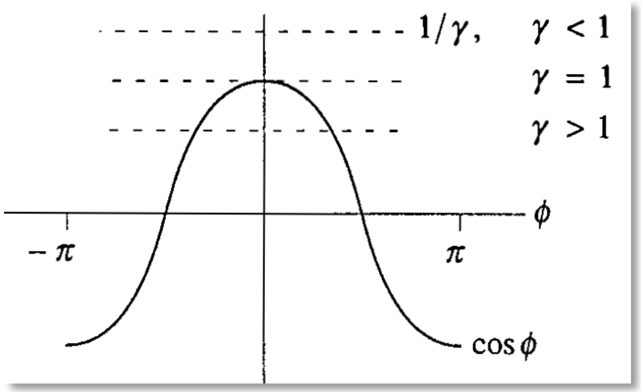
- Fixed points from: sin ϕ = 0

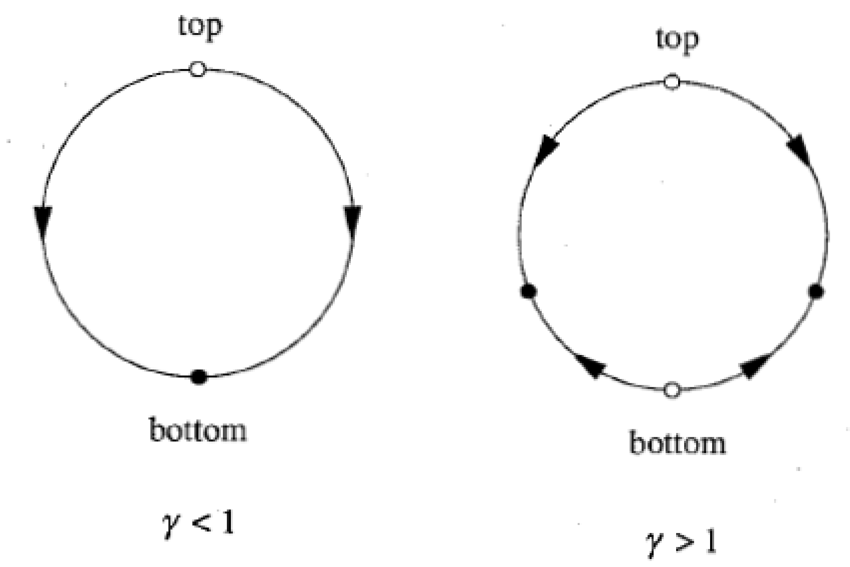
$\phi^* = 0$ (the bottom of the hoop) and $\phi^* = \pi$ (the top).

stable unstable

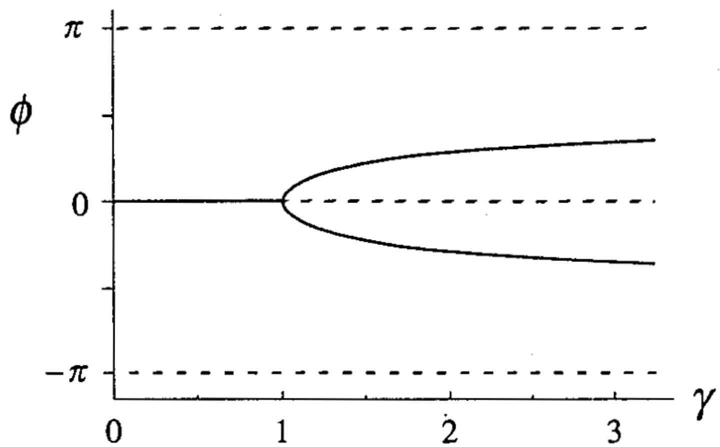
- Fixed points from: γ cos ϕ -1 = 0

$$\gamma = \frac{r\omega^2}{g}$$





Bifurcation diagram:



When is this "first-order" description valid?

When is ok to neglect the second derivative d^2x/dt^2 ?

Dimensional analysis and scaling

Dimensionless equation

- Dimensionless time
  (T = characteristic time-scale) $\tau = \frac{t}{T}$

$$\left(\frac{r}{gT^2}\right)\frac{d^2\phi}{d\tau^2} = -\left(\frac{b}{mgT}\right)\frac{d\phi}{d\tau} - \sin\phi + \left(\frac{r\omega^2}{g}\right)\sin\phi\cos\phi$$

- We want the lhs very small, we define T such that

$$\frac{r}{gT^2} << 1 \quad \text{and} \quad \frac{b}{mgT} \approx O(1) \quad \Rightarrow \quad T = \frac{b}{mg}$$

$$\frac{r}{g}\left(\frac{mg}{b}\right)^2 << 1 \quad \Rightarrow \quad \boxed{b^2 >> m^2 gr}$$

- Define: $\varepsilon = \frac{m^2 gr}{b^2} \Rightarrow \boxed{\varepsilon\frac{d^2\phi}{d\tau^2} = -\frac{d\phi}{d\tau} - \sin\phi + \gamma\sin\phi\cos\phi}$ $\quad \gamma = \frac{r\omega^2}{g}$

Over damped limit

$$\varepsilon \frac{d^2\phi}{d\tau^2} = -\frac{d\phi}{d\tau} - \sin\phi + \gamma \sin\phi \cos\phi$$

- The dimension less equation suggests that the first-order equation is valid in the over damped limit: $\varepsilon \to 0$
- Problem: second-order equation has two independent initial conditions: $\phi(0)$ and $d\phi/d\tau(0)$
- But the first-order equation has only one initial condition $\phi(0)$, $d\phi/d\tau(0)$ is calculated from

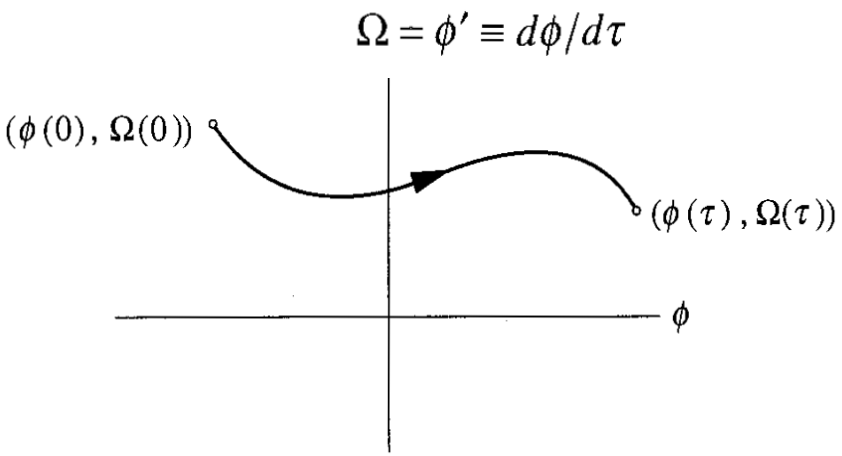
$$\frac{d\phi}{d\tau} = -\sin\phi + \gamma \sin\phi \cos\phi$$

- Paradox: how can the first-order equation represent the second-order equation?

Trajectories in phase space

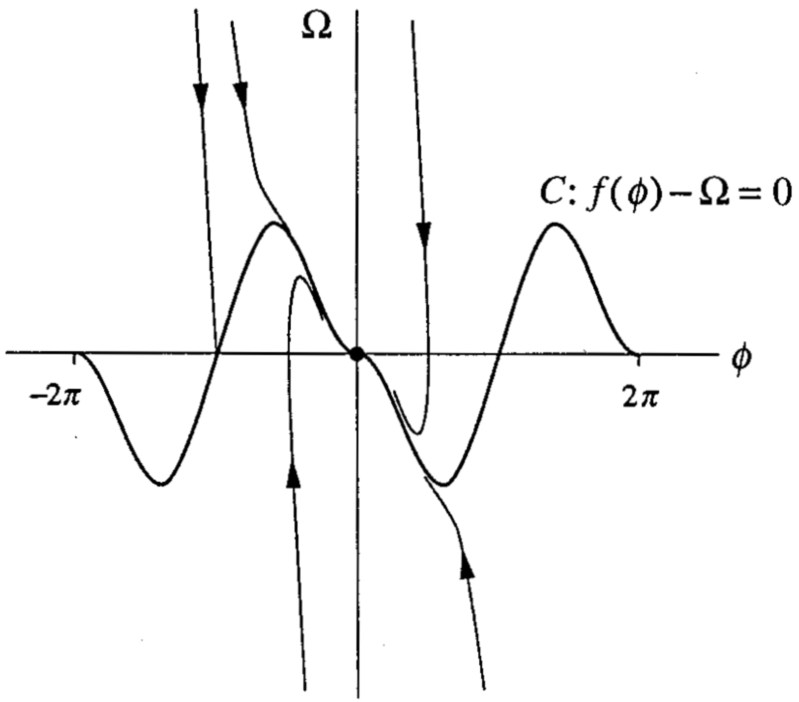
- First order system:

$$\frac{d\phi}{d\tau} = f(\phi) - \sin\phi + \gamma \sin\phi \cos\phi$$



- Second order system:

$$\varepsilon \frac{d^2\phi}{d\tau^2} = -\frac{d\phi}{d\tau} - \sin\phi + \gamma \sin\phi \cos\phi$$



Second order system:

$\varepsilon \to 0$

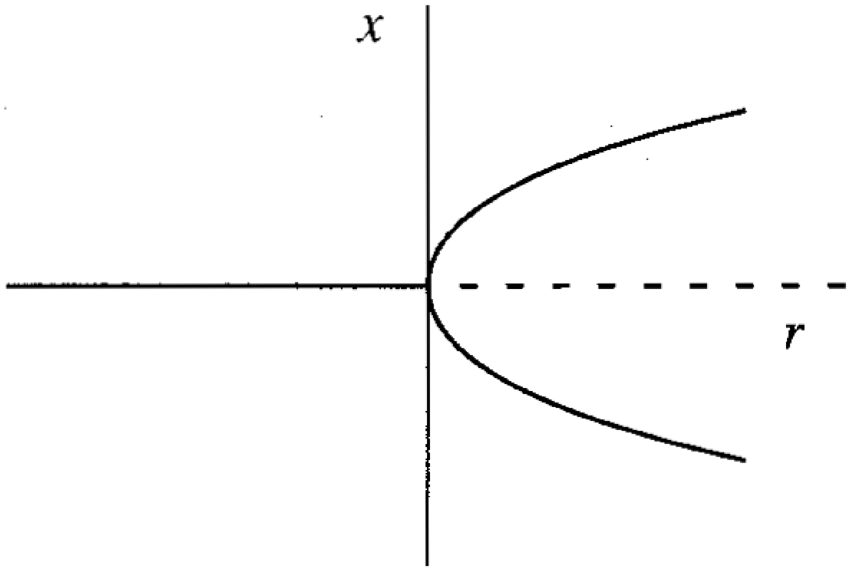
limit, *all trajectories slam straight up or down onto the curve C defined by $f(\phi) = \Omega$, and then slowly ooze along this curve until they reach a fixed point*

Outline

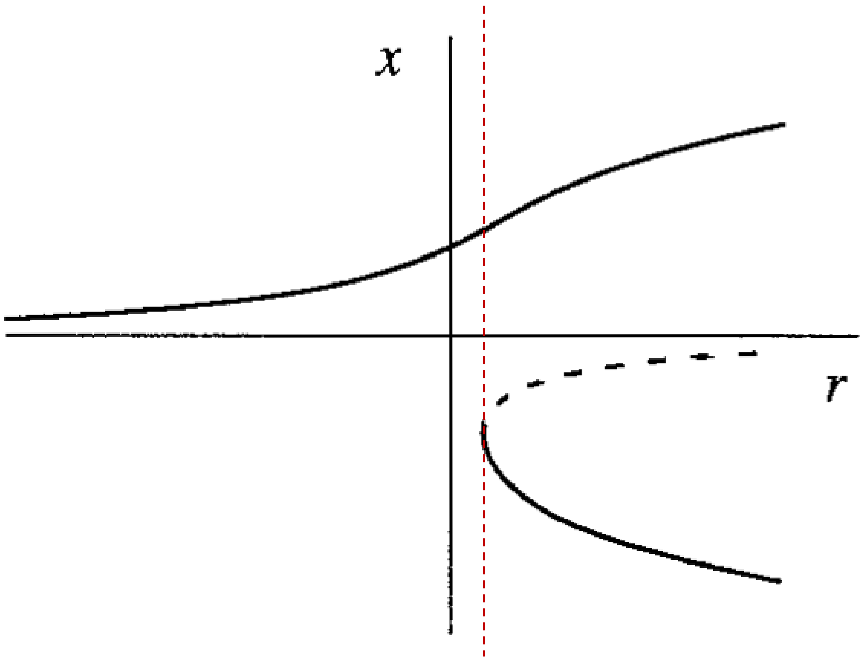
- Introduction to bifurcations
- Saddle-node, transcritical and pitchfork bifurcations
- Examples
  - Saddle-node: neuron model
  - Transcritical: laser threshold
  - Pitchfork: particle in a rotating wire hoop
- Imperfect bifurcations & catastrophes
  - Example: insect outbreak
- Delayed feedback examples

Imperfect bifurcations

$$\dot{x} = h + rx - x^3$$



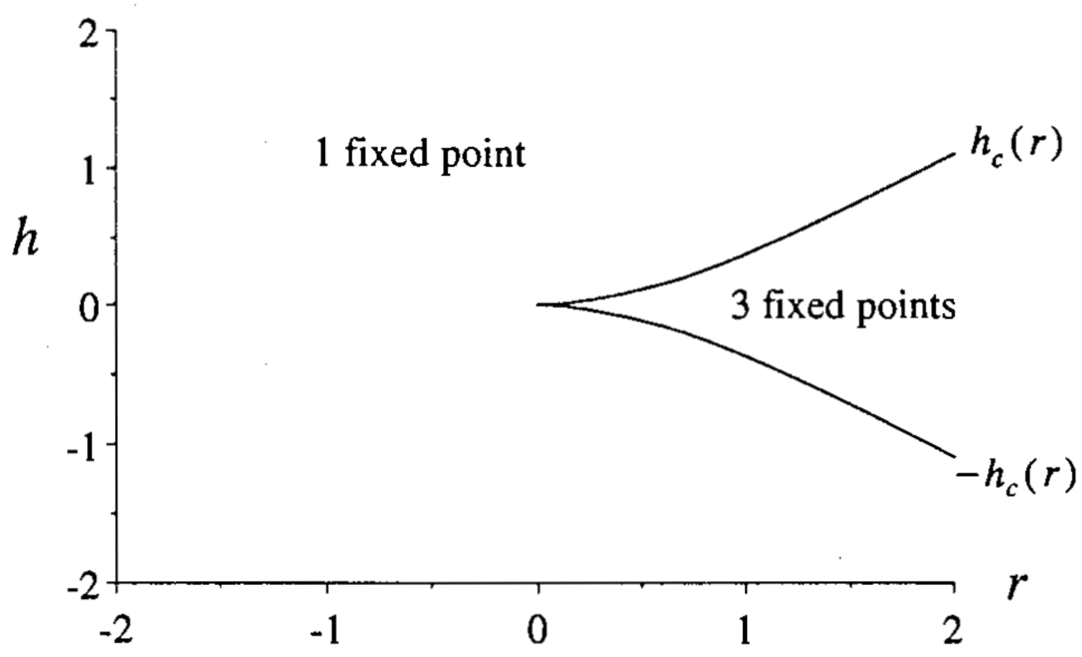
(a) $h = 0$



(b) $h \neq 0$

Parameter space (*h*, *r*)

$$\dot{x} = h + rx - x^3$$



Exercise : using these two equations

1. fixed points: $f(x^*) = 0$
2. saddle node bifurcation: $f'(x^*) = 0$

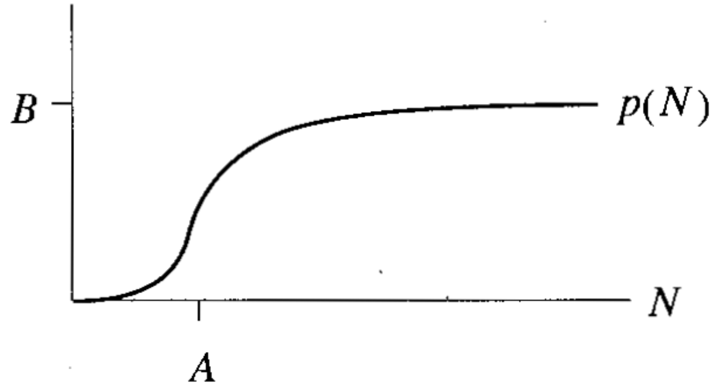
Calculate $h_c(r)$

$$h_c = \pm \frac{2r}{3}\sqrt{\frac{r}{3}}$$

Example: insect outbreak

$$\dot{N} = RN\left(1 - \frac{N}{K}\right) - p(N)$$

- Budworms population grows logistically (*R*>0 grow rate)
- *p*(*N*): dead rate due to predation
- If no budworms (*N*≈0): no predation: birds look for food elsewhere
- If *N* large, *p*(*N*) saturates: birds eat as much as they can.



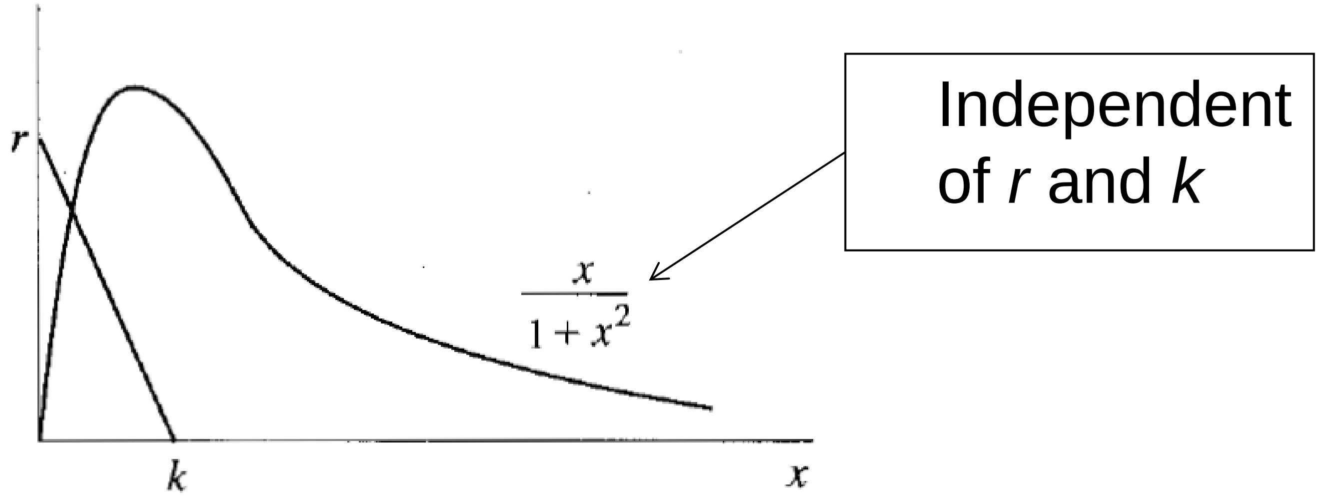
$$p(N) = \frac{BN^2}{A^2 + N^2} \qquad A, B > 0$$

Dimensionless formulation

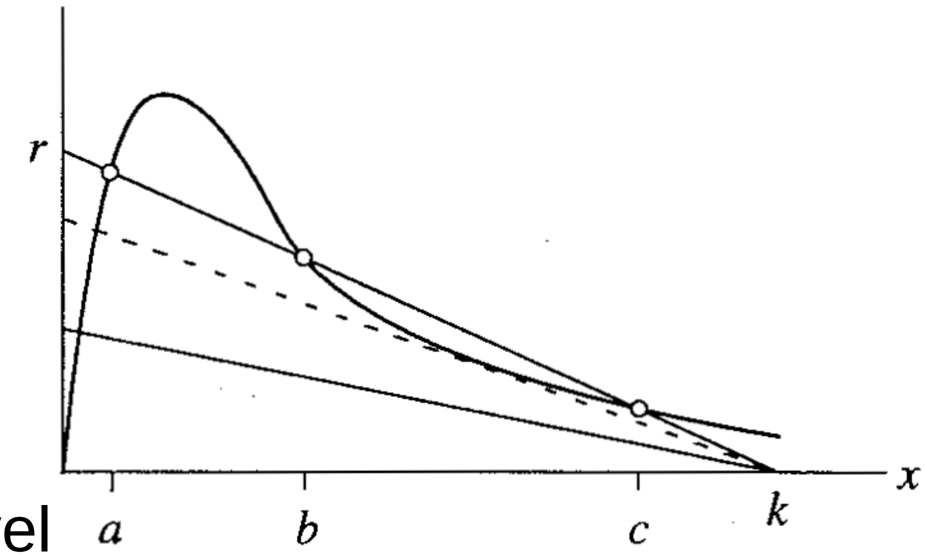
$$x = N/A \qquad \tau = \frac{Bt}{A}, \qquad r = \frac{RA}{B}, \qquad k = \frac{K}{A}$$

$$\frac{dx}{d\tau} = rx\left(1 - \frac{x}{k}\right) - \frac{x^2}{1+x^2}$$

- $x^*=0$
- Other FPs from the solution of $r\left(1 - \frac{x}{k}\right) = \frac{x}{1+x^2}$



- When the line intersects the curve tangentially (dashed line): saddle-node bifurcation



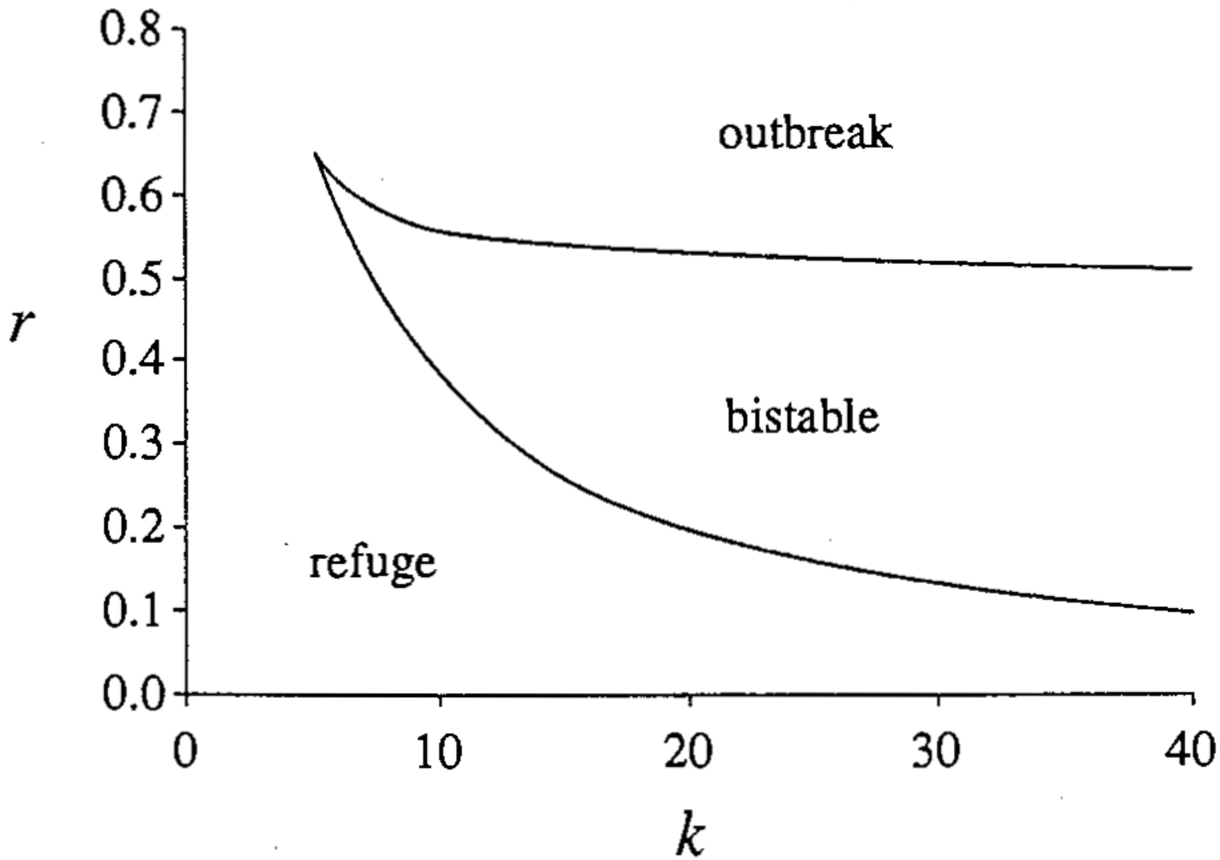
a: Refuge level of the budworm population

b: threshold

c: Outbreak level (pest)

Exercise : show that x*=0 is always unstable

Parameter space (*k*, *r*)



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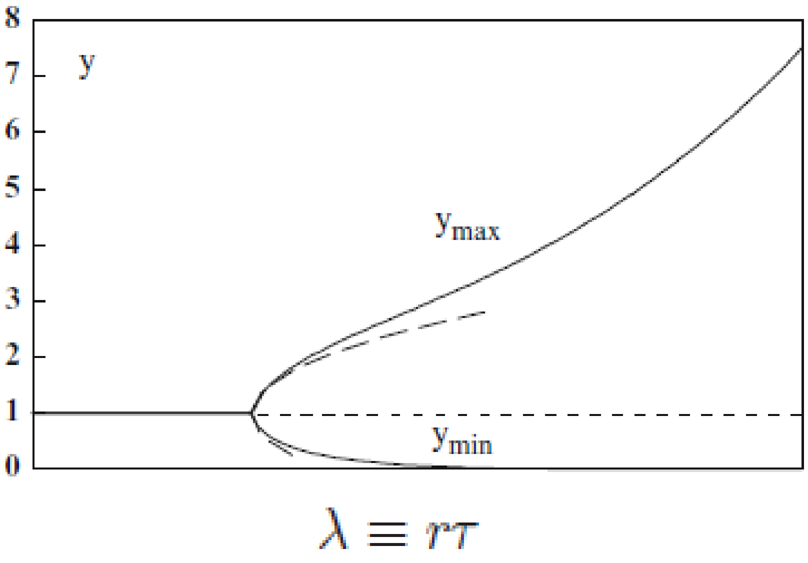
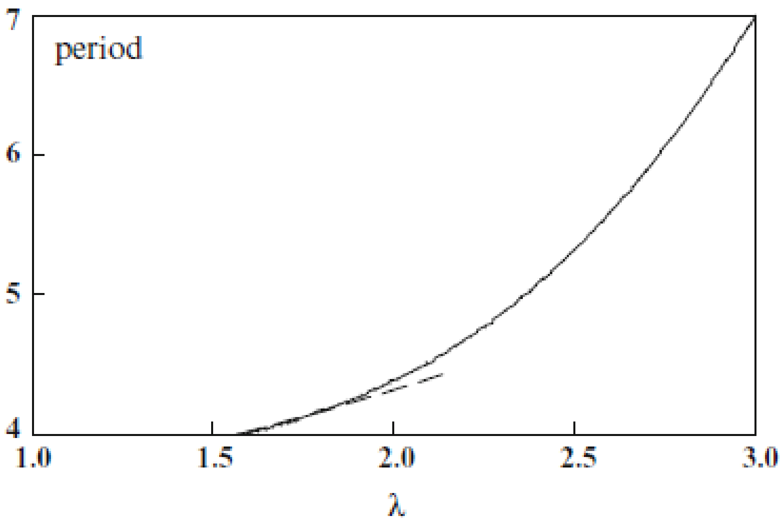
Bifurcations in 1D systems with delay
Example 1: delayed logistic equation

$$\frac{dN}{dt'} = rN(1 - \frac{N(t' - \tau)}{K}) \qquad \frac{dy}{dt} = \lambda y(1 - y(t-1)) \qquad \lambda \equiv r\tau$$

- Delay allows for sustained oscillations in a single species population, without any predatory interaction of other species





- Hopf bifurcation (more in Part 2 = 2D dynamical systems)

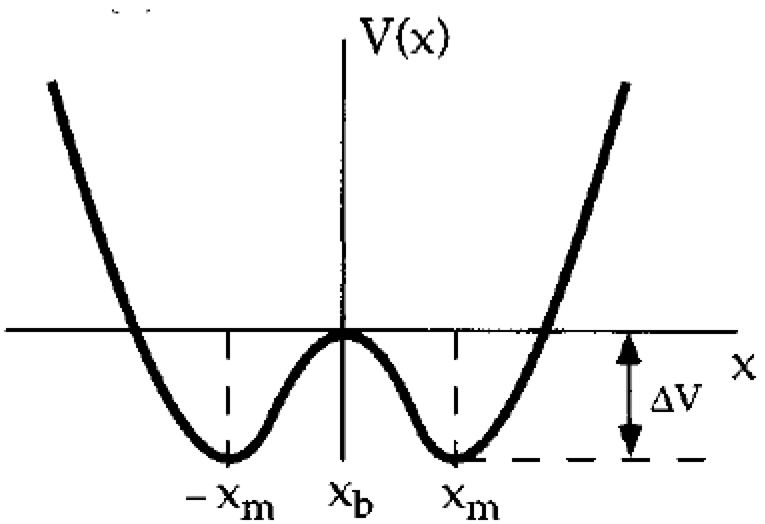
Example 2: particle in a double-well potential with delayed feedback

$$\frac{dx}{dt} = x - x^3 + c\,x(t-\tau) + \sqrt{2D}\xi$$

noise

$$= -\frac{\partial V}{\partial x} + c\,x(t-\tau) + \sqrt{2D}\xi$$

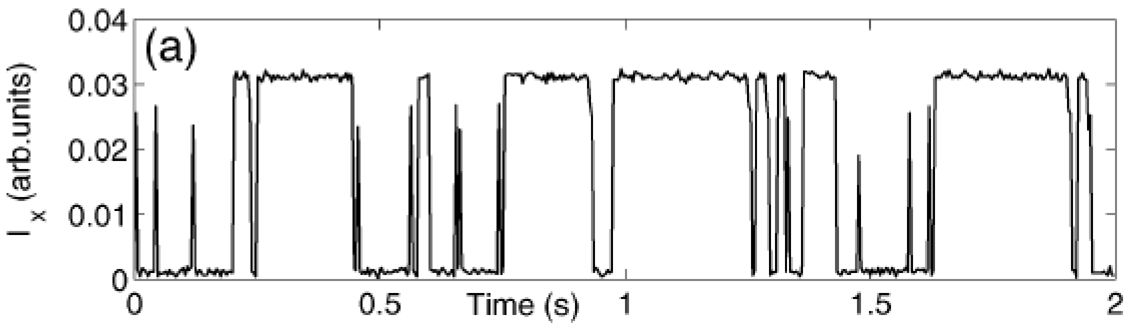
$$V = -x^2/2 + x^4/4$$



Motivation

$$\frac{dx}{dt} = x - x^3 + c\,x(t-\tau) + \sqrt{2D}\xi$$

- Simple model to understand two-state systems
- Example: in the light emitted by a laser with feedback, observation of switching between X and Y polarization.



- With appropriated parameters delay feedback can control the movement and confine the system in one state.

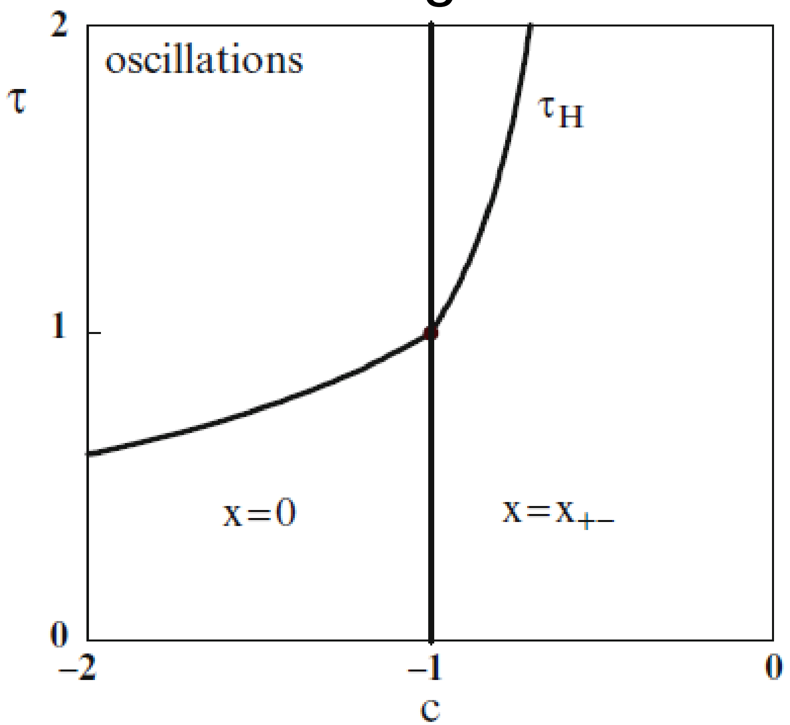
Fixed point solutions (deterministic equation, D=0)

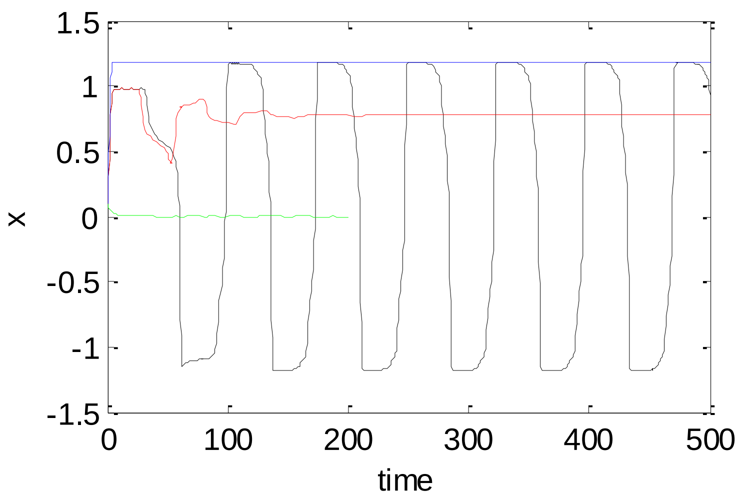
$$x = 0,$$

$$x = x_{\pm} \equiv \pm\sqrt{1+c} \;(c \geq -1)$$

if c>0 stable for all τ
if c<0 the stability depends on (c, τ)

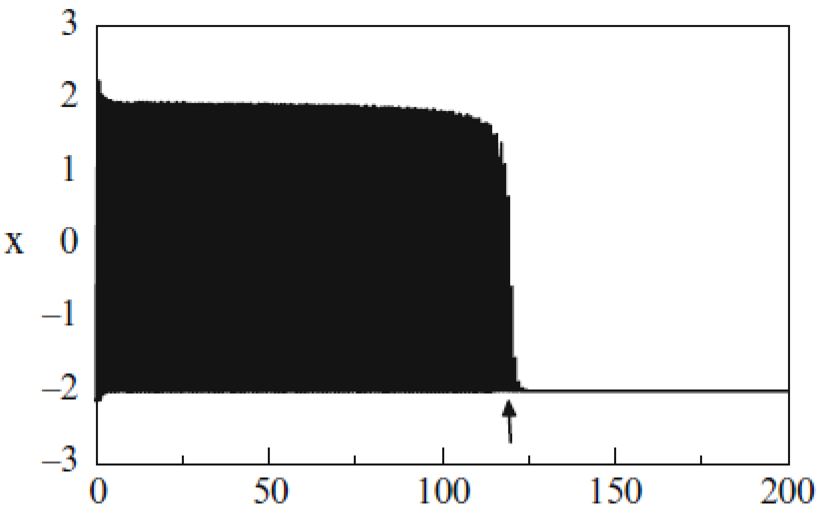
- Phase Diagram





blue: c=0.4, τ=1
red: c=-0.4, τ=25
black: c=-0.4, τ=30
green: c=-1.1, τ=0.5

Special initial conditions give meta-stability and long transients



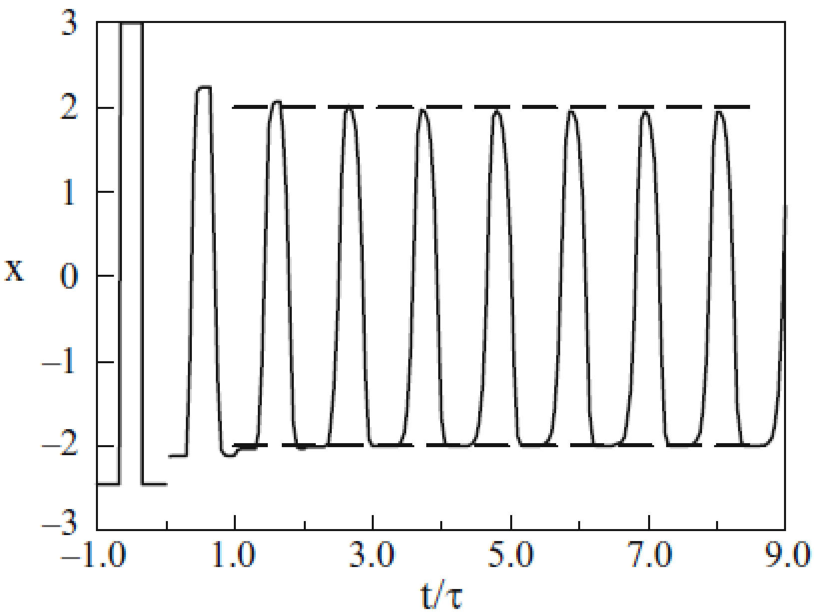


Figure 2.10: Top: slowly varying oscillations followed by a sudden jump to the steady-state $x = -2$. Bottom: short time solution showing the initial conditions: $x = -2.45$ ($-\tau < t < -2\tau/3$ and $-\tau/3 < t < 0$) and $x = 3$ ($-2\tau/3 < t < -\tau/3$). The values of the parameters are $c = 3$ and $\tau = 5$.

Class/Homework

3.5.4 (Bead on a horizontal wire) A bead of mass m is constrained to slide along a straight horizontal wire. A spring of relaxed length L_0 and spring constant k is attached to the mass and to a support point a distance h from the wire (Figure 1).

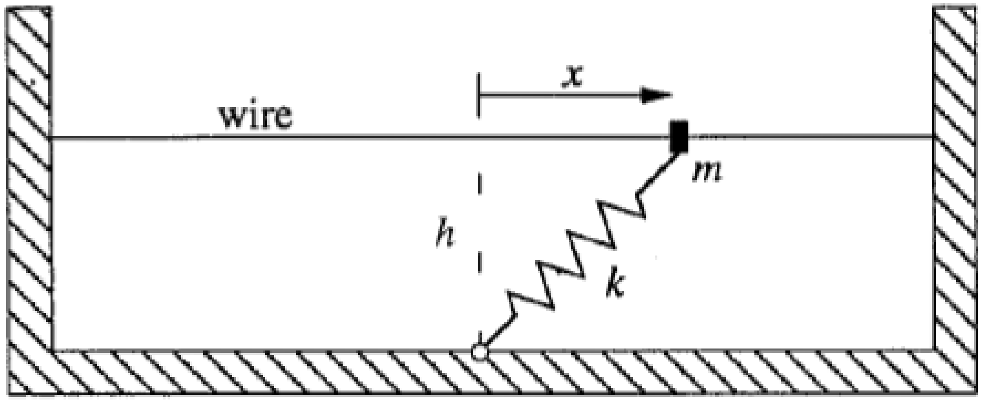


Figure 1

Finally, suppose that the motion of the bead is opposed by a viscous damping force $b\dot{x}$.

a) Write Newton's law for the motion of the bead.

b) Find all possible equilibria, i.e., fixed points, as functions of k, h, m, b, and L_0.

c) Suppose $m = 0$. Classify the stability of all the fixed points, and draw a bifurcation diagram.

d) If $m \neq 0$, how small does m have to be to be considered negligible? In what sense is it negligible?

Class/Homework

3.7.6 (Model of an epidemic) In pioneering work in epidemiology, Kermack and McKendrick (1927) proposed the following simple model for the evolution of an epidemic. Suppose that the population can be divided into three classes: $x(t)$ = number of healthy people; $y(t)$ = number of sick people; $z(t)$ = number of dead people. Assume that the total population remains constant in size, except for deaths due to the epidemic. (That is, the epidemic evolves so rapidly that we can ignore the slower changes in the populations due to births, emigration, or deaths by other causes.)

Then the model is

$$\dot{x} = -kxy$$
$$\dot{y} = kxy - \ell y$$
$$\dot{z} = \ell y$$

where k and ℓ are positive constants. The equations are based on two assumptions:

(i) Healthy people get sick at a rate proportional to the product of x and y. This would be true if healthy and sick people encounter each other at a rate proportional to their numbers, and if there were a constant probability that each such encounter would lead to transmission of the disease.

(ii) Sick people die at a constant rate ℓ.

The goal of this exercise is to reduce the model, which is a *third-order system*, to a first-order system that can analyzed by our methods. (In Chapter 6 we will see

a) Show that $x + y + z = N$, where N is constant.

b) Use the $\dot{x}$ and $\dot{z}$ equation to show that $x(t) = x_0 \exp(-kz(t)/\ell)$, where $x_0 = x(0)$.

c) Show that z satisfies the first-order equation $\dot{z} = \ell\left[N - z - x_0 \exp(-kz/\ell)\right]$.

d) Show that this equation can be nondimensionalized to

$$\frac{du}{d\tau} = a - bu - e^{-u}$$

by an appropriate rescaling.

e) Show that $a \geq 1$ and $b > 0$.

f) Determine the number of fixed points u^* and classify their stability.

g) Show that the maximum of $\dot{u}(t)$ occurs at the same time as the maximum of both $\dot{z}(t)$ and $y(t)$. (This time is called the ***peak*** of the epidemic, denoted t_{peak}. At this time, there are more sick people and a higher daily death rate than at any other time.)

h) Show that if $b < 1$, then $\dot{u}(t)$ is increasing at $t = 0$ and reaches its maximum at some time $t_{\text{peak}} > 0$. Thus things get worse before they get better. (The term ***epidemic*** is reserved for this case.) Show that $\dot{u}(t)$ eventually decreases to 0.

Bibliography

- Steven H. Strogatz: *Nonlinear dynamics and chaos, with applications to physics, biology, chemistry and engineering* (Addison-Wesley Pub. Co., 1994). Chapter 3

- Thomas Erneux: *Applied delay differential equations* (Springer 2009).

- Thomas Erneux and Pierre Glorieux: *Laser Dynamics* (Cambridge University Press 2010)

- Eugene M. Izhikevich: *Dynamical Systems in Neuroscience: The Geometry of Excitability and Bursting* (MIT Press 2010)