Nonlinear systems, chaos and control in Engineering

Module 1

One-dimensional systems

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Schedule

Flows on the line

(Strogatz ch.1 & 2)

18/11 (3 hs)

- Introduction
- Solving equations with computer
- Fixed points and linear stability
- Feedback control and delays

Bifurcations

(Strogatz ch. 3)

18/11 (2 hs) & 24/11 (3 hs)

- Introduction
- Saddle-node
- Transcritical
- Pitchfork
- Examples

Flows on the circle

(Strogatz ch. 4)

25/11 (2 hs)

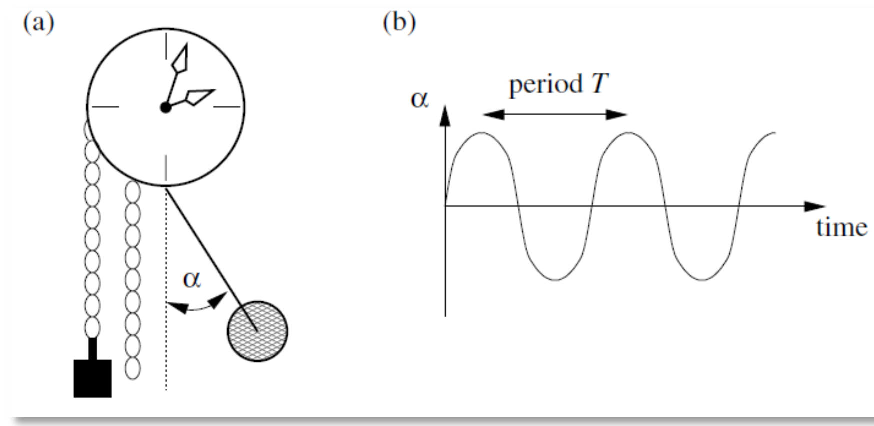
- Introduction to phase oscillators
- Nonlinear oscillator
- Fireflies and entrainment

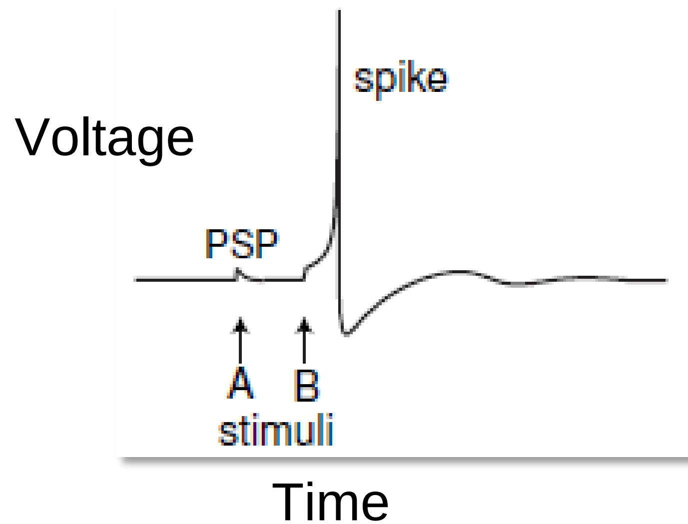
Flows on the line: outline

- **Introduction to dynamical systems**
- Introduction to flows on the line
- Solving equations with computer
- Fixed points and linear stability
- Feedback control: delay differential equations

Dynamical Systems

- Systems that evolve in time.
- Examples:
  - Pendulum clock
  - Neuron
- Dynamical systems can be:
  - *linear or nonlinear (harmonic oscillator – pendulum);*
  - *deterministic or stochastic;*
  - *low or high dimensional;*
  - *continuous time or discrete time.*





In this course: nonlinear systems (**Nonlinear Dynamics**)

A neuron: an stochastic nonlinear dynamical system

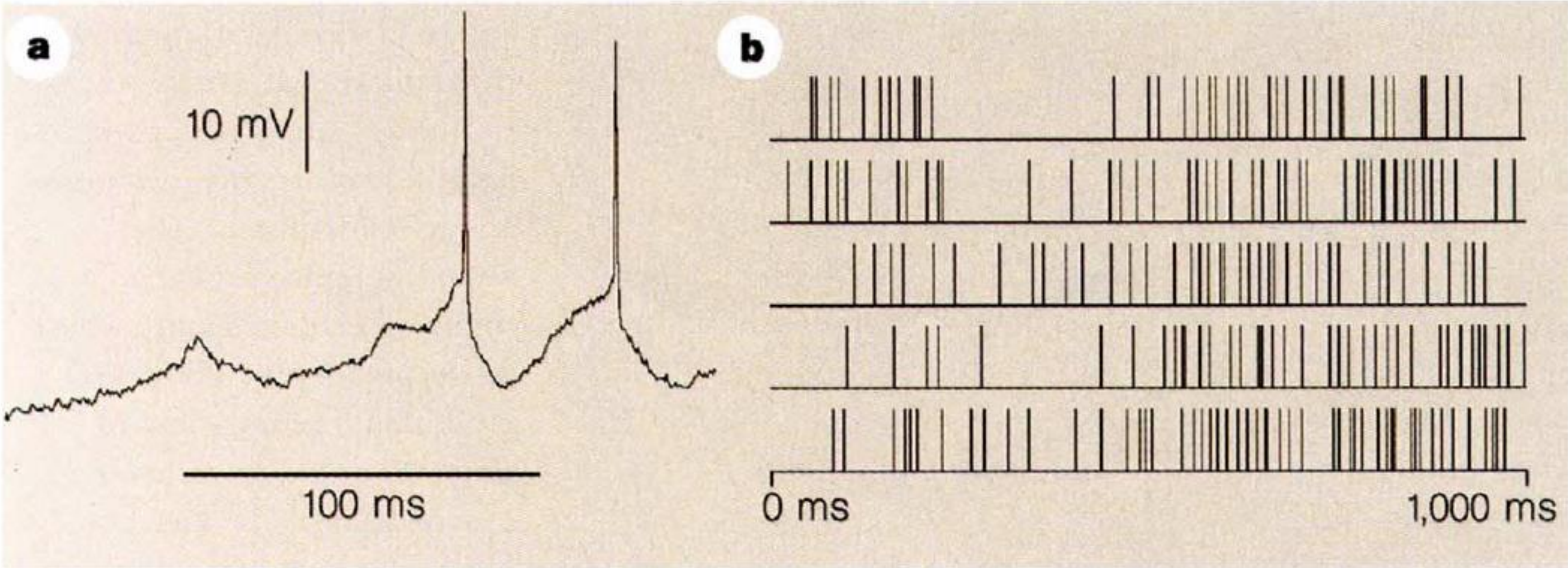
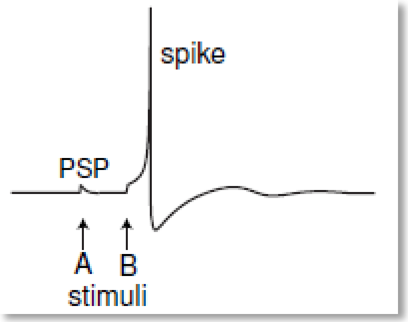


Figure 2 a, **The membrane potential of a single neuron in the cat visual cortex. A visual stimulus causes a sustained barrage of synaptic input, which triggers three cycles of depolarization. The first cycle does not reach the threshold for generating a spike, but the second and third do. (Data provided by B. Ahmed and K. Martin.)** b, **Variability of the firing response of a single neuron in monkey cortex (each line in the trace corresponds to a spike of the type shown in** a**). The same stimulus is presented five times and triggers about 40 spikes each time; yet the exact timing of individual spikes shows random variation. (Data provided by W. Newsome and K. Britten.)**

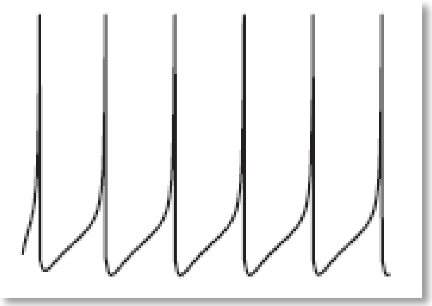
Koch Nature1997

Given the initial condition: possible evolution

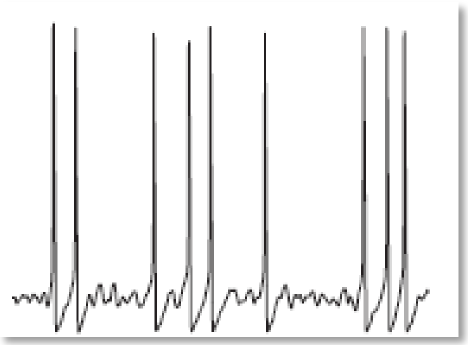
- The neuron settles down to equilibrium (rest state or "fixed point"). – module 1 in this course



- Keeps spiking in cycles ("limit cycle"). – module 2



- More complicated: **chaotic** or **complex** evolution ("chaotic attractor"). – module 3



Historical development of the *Theory of Dynamical Systems*

mid-1600s: Ordinary differential equations (ODEs)

- **Isaac Newton**: studied planetary orbits and solved analytically the "two-body" problem (earth around the sun).

- Since then: a lot of effort for solving the "three-body" problem (earth-sun-moon) – Impossible.

Christiaan Huygens (mid-1600s, Dutch mathematician)

- Patented the first pendulum clock.
- Observed the synchronization of two clocks.

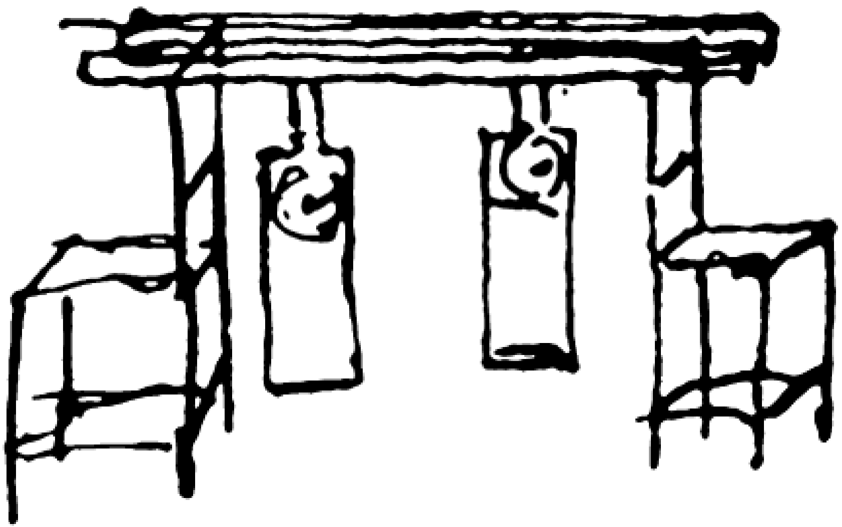
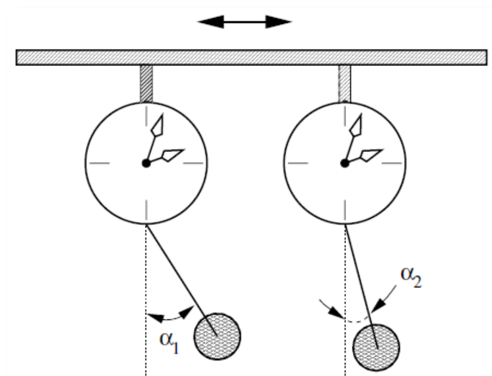


Figure 1.2. Original drawing of Christiaan Huygens illustrating his experiments with two pendulum clocks placed on a common support.



http://www.youtube.com/watch?v=izy4a5erom8

Late 1800s

- **Henri Poincare** (French mathematician).

  Instead of asking "*which are the exact positions of planets (trajectories)*?"

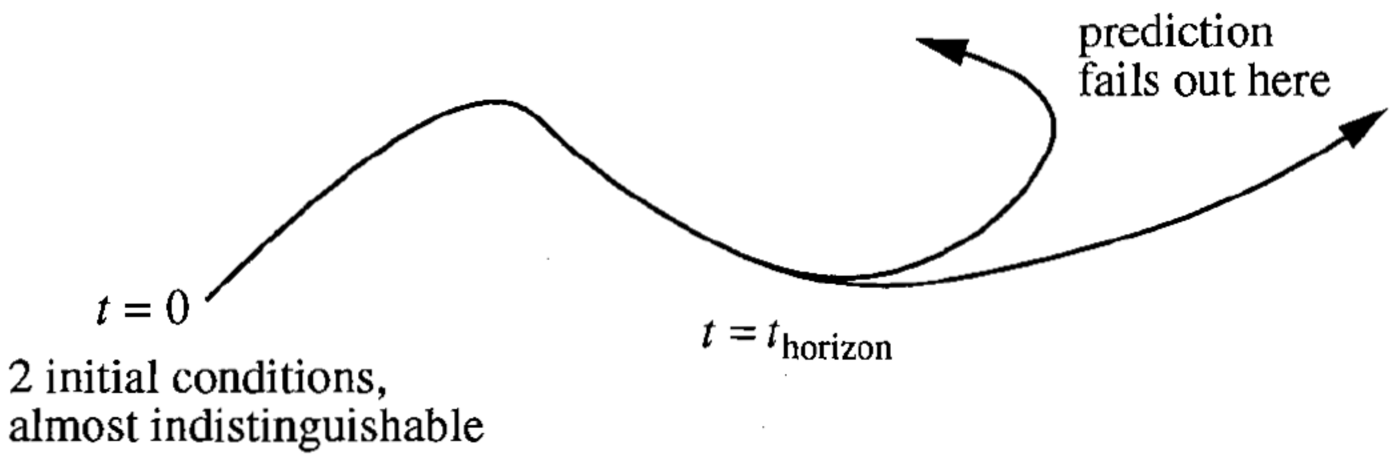
  he asked: "*is the solar system* ***stable*** *for ever, or will planets eventually run away*?"

- He developed a **geometrical** approach to solve the problem.

- Introduced the concept of "phase space".

- He also had an intuition of the possibility of **chaos**:

Poincare: "The evolution of a ***deterministic*** *system can be aperiodic, unpredictable, and strongly depends on the initial conditions"*



Deterministic system: the present state (initial condition) fully determines the future state. There is no randomness but the system can be unpredictable.

1950s: First simulations

- Computers drive economic growth and transform how we live and work.
- Computes allowed to experiment with equations.
- Powerful tool to advance the "*Theory of Dynamical Systems*".
- 1960s: **Eduard Lorentz** (American mathematician and meteorologist at MIT): simple model of convection rolls in the atmosphere.
- Intuition of **chaotic** motion on a **strange** attractor.
- He also showed that there is **structure** and **order** in chaotic motion.

The Lorentz system

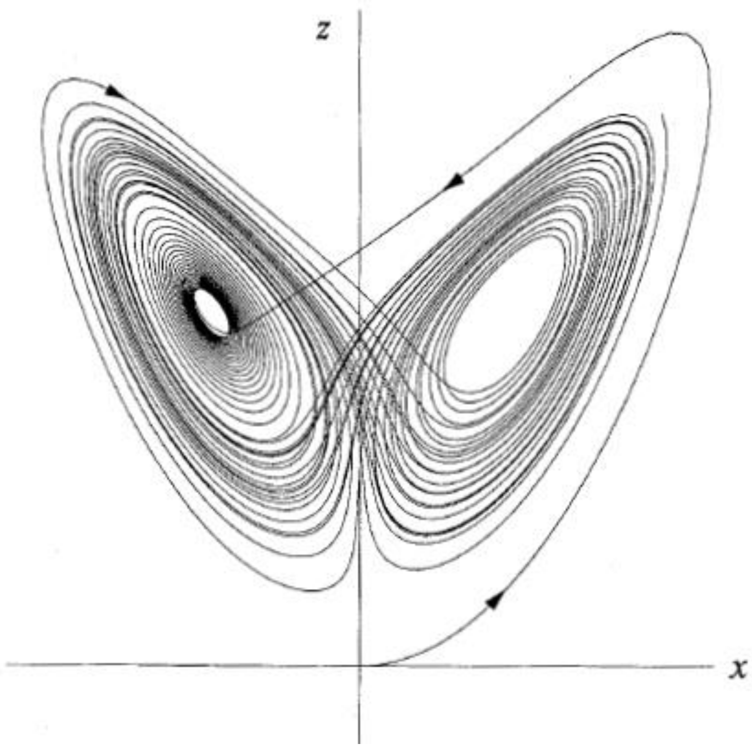
Lorentz studied meteorological prediction using Navier-Stokes simplified equations:

$$\dot{x} = \delta(x - y)$$
$$\dot{y} = x(r - z) - y$$
$$\dot{z} = xy - bz$$

- 3 variables:
  - x: rotation rate of a cylindrical mass of gas,
  - y: thermal gradient,
  - z: temperature variation.
- 3 Parameters:
  - δ: ratio between viscosity and thermal conductivity (Prandtl number),
  - R: temperature difference between top and bottom of cylinder (Rayleigh number),
  - b: ratio between width and height of the cylinder.

Lorentz's Attractor

- Starting from an initial condition (x_0, y_0, z_0) by numerically integrating the equations we can plot the trajectory in the *phase space* (Lorentz's Attractor).



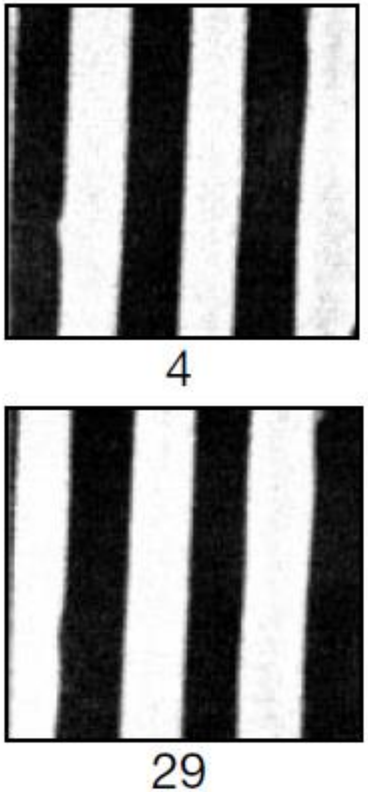
Lorentz found extreme sensitivity to initial conditions

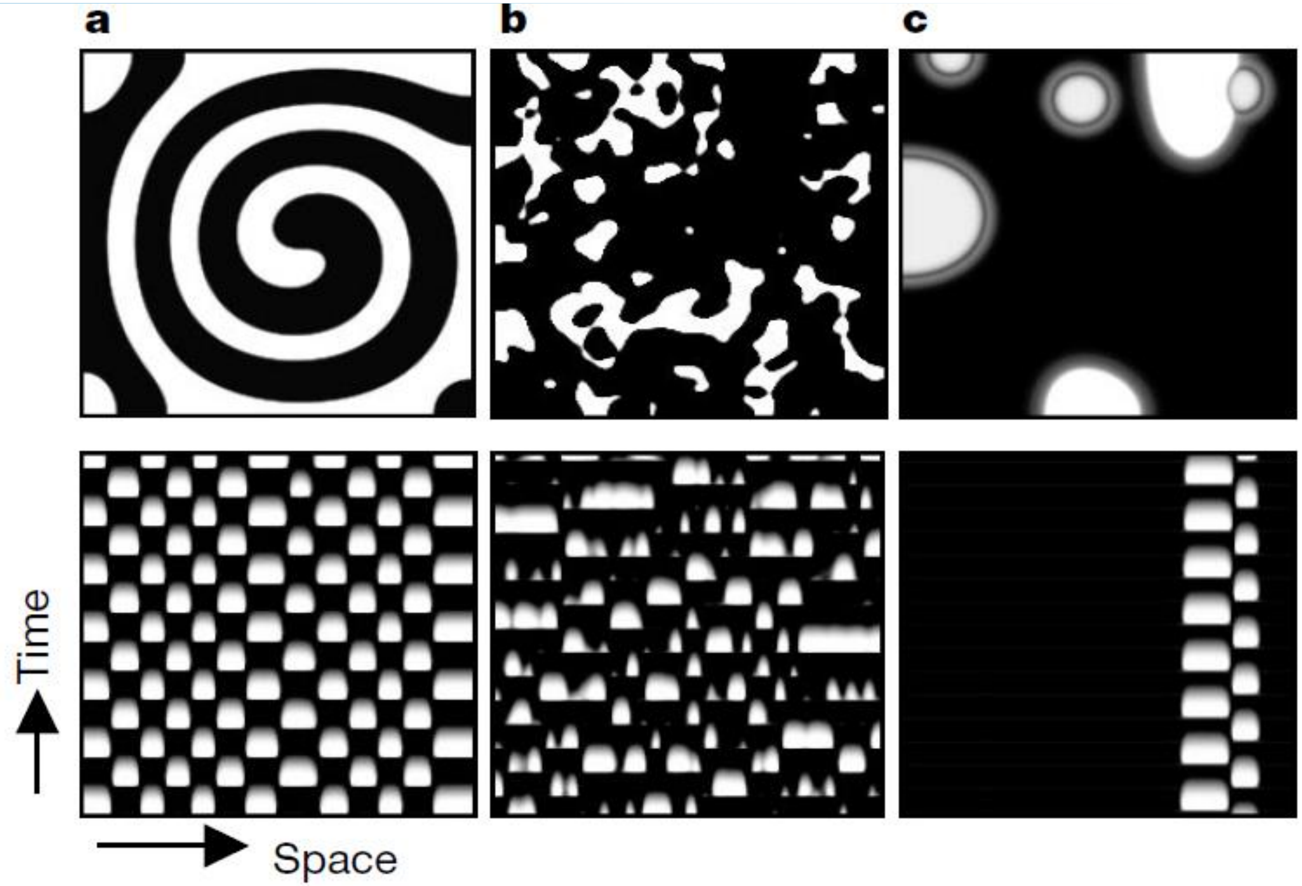
$\Rightarrow$ impossibility of long-term meteorological predictions.

Order within chaos and self-organization

- **Ilya Prigogine** (Belgium, born in Moscow, Nobel Prize in Chemistry 1977)
- Thermodynamic systems far from equilibrium.
- Discovered that, in chemical systems, the interplay of (external) **input of energy** and **dissipation** can lead to "self-organised" patterns.
- Reverse the rule of maximization of entropy (second law of thermodynamics).
- Wide implications to biological systems and the evolution of life.

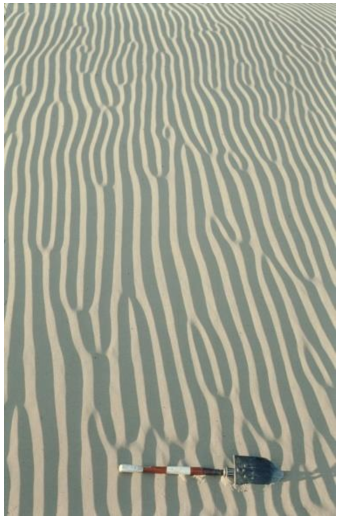
One-dimensional spatio-temporal patterns

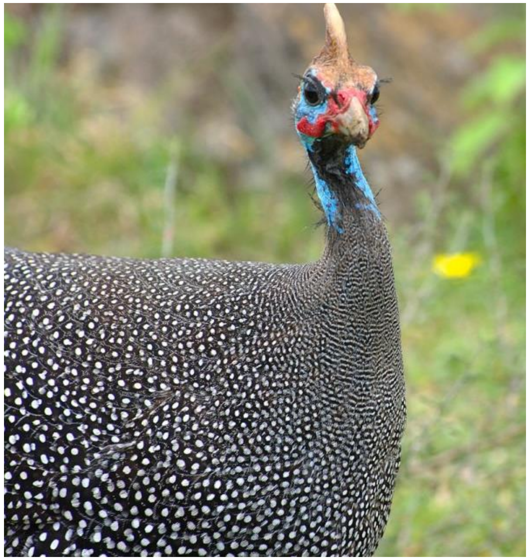
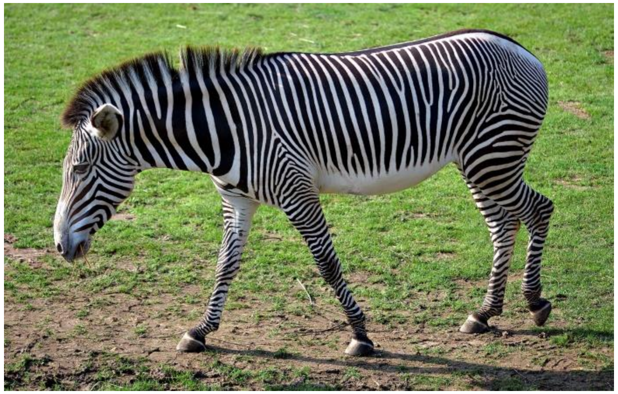


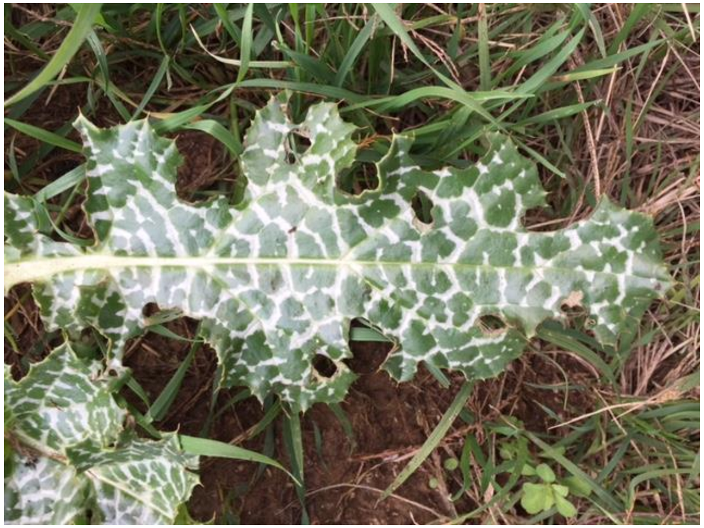


Vanag et al Nature 2000

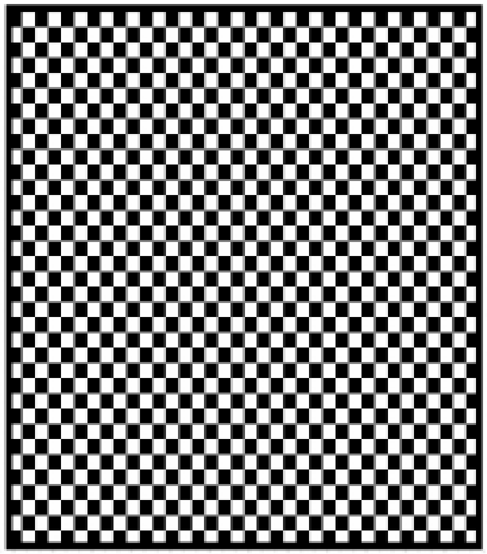
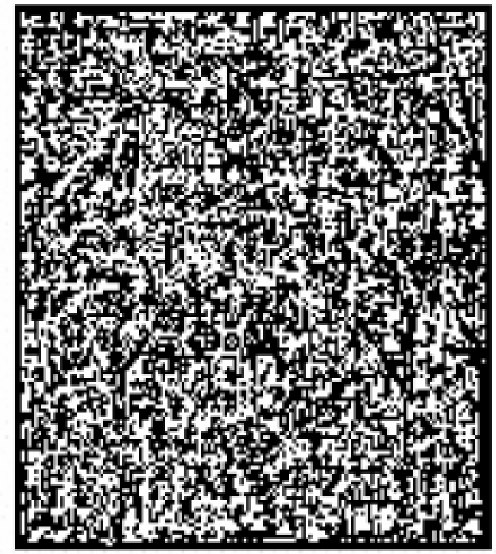
Patterns in nature
(source: wikipedia)



Entropy (H) and Complexity (C)

Complete order	Chaos	Complete disorder
		
$H = 0$ $C = 0$	$H \neq 0$ $C \neq 0$	$H = Max$ $C = 0$

O. A. Rosso (2009)

According to Prigogine

Newtonian physics has been extended three times:

- First, with the use of the wave function in quantum mechanics.
- Then, with the introduction of space-time in relativity.
- And finally, with the recognition of indeterminism in nonlinear systems.

Chaos is the third great revolution of 20th-century physics*, after relativity and quantum theory.*

In the 1960s: biological nonlinear oscillators

- **Arthur Winfee** (American theoretical biologist – born in St. Petersburg): Large communities of biological oscillators show a tendency to self-organize in time –**collective synchronization**.

In the 1960's he did experiments trying to understand the effects of perturbations in biological clocks (circadian rhythms).

What is the effect of an external perturbation on subsequent oscillations?

Winfee work on perturbing biological oscillators

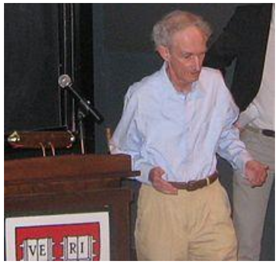
- He studied the periodic emergence of a fruit fly that as a ≈24-hour rhythmic emergency.

- Using brief pulses of light, found that the periodic emergence of the flies was shifted, and the shift depended on the timing and the duration of the light pulse.

- Also found that there is a critical timing and duration that results in no further periodic emergency (destroys the biological clock).

- The work has wide implications, for example, for cardiac tissue: some cardiac failures are related to perturbed oscillations.

The 1970s



- **Robert May** (Australian, 1936): population biology
- "Simple mathematical models with very complicated dynamics", *Nature* (1976).

$$x_{t+1} = f(x_t)$$ Example: $$f(x) = r\,x(1-x)$$

- Difference equations ("iterated maps"), even though simple and deterministic, can exhibit different types of dynamical behaviors, from **stable points**, to a bifurcating hierarchy of **stable cycles**, to **apparently random fluctuations**.

Models of dynamical systems

Continuous-time ordinary differential equations (ODEs)

$$\frac{dx}{dt} = \dot{x} = f(x, r)$$

Discrete-time equations (iterated maps)

$$x_{t+1} = f(x_t, r)$$

r = control parameter(s)

Example: Lorentz model

$$\dot{x} = \delta(x - y)$$
$$\dot{y} = x(r - z) - y$$
$$\dot{z} = xy - bz$$

r = (δ, r, b)

Example: Logistic map

$$x_{t+1} = r\, x_t (1 - x_t)$$

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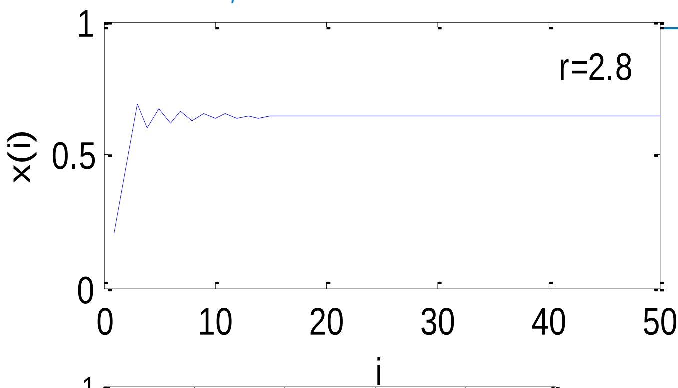
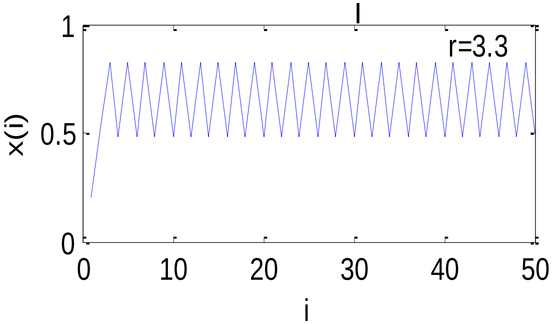
The logistic map

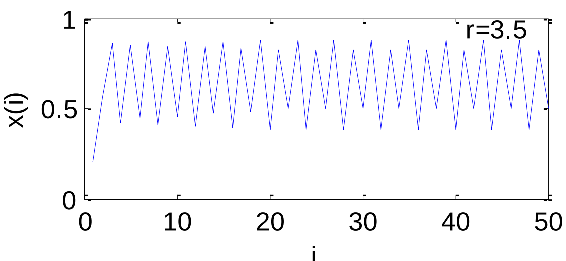
$$x(i+1) = r\,x(i)[1-x(i)]$$

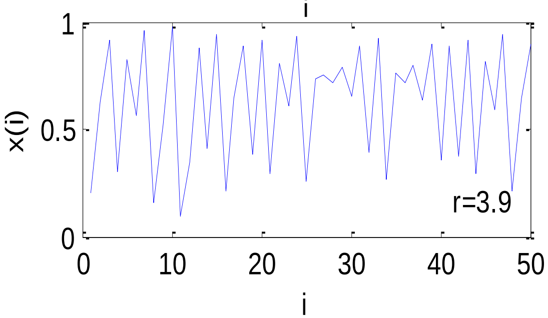


r=2.8, Initial condition: x(1) = 0.2

Transient relaxation → long-term stability

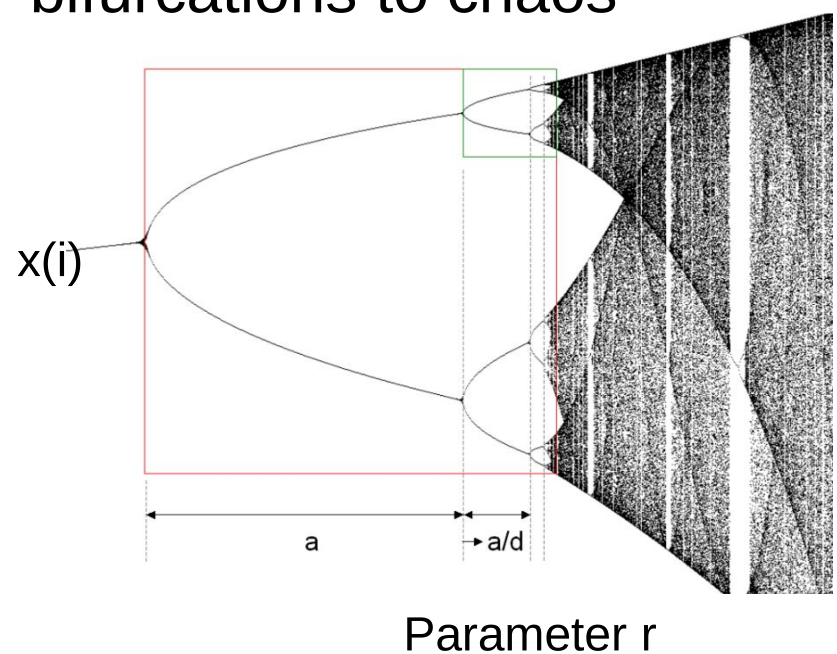






Transient dynamics → stationary oscillations (regular or irregular)

"period-doubling" bifurcations to chaos



Universal route to chaos

- In 1975, **Mitchell Feigenbaum** (American mathematical physicist), using a small HP-65 calculator, discovered the scaling law of the bifurcation points

$$\lim_{n\to\infty} \frac{r_{n-1} - r_{n-2}}{r_n - r_{n-1}} = 4.6692...$$

- Then, he provided a mathematical proof (by using the "renormalization concept" –connecting to phase transitions in statistical physics).
- Then, he showed that the same behavior, with the same mathematical constant, occurs within a wide class of functions, prior to the onset of chaos (**universality**).

Very different systems (in chemistry, biology, physics, etc.) go to chaos in the same way, quantitatively.

HP-65 calculator

- The first magnetic card-programmable handheld calculator

HP-65 in original hard case with manuals, software "Standard Pac" of magnetic cards, soft leather case, and charger

The late 1970s

- **Benoit Mandelbrot** (Polish-born, French and American mathematician 1924-2010): "self-similarity" and **fractal objects:**

  each part of the object is like the whole object but smaller.

- Because of his access to IBM's computers, Mandelbrot was one of the first to use **computer graphics** to create and display fractal geometric images.

Fractal objects

- Are characterized by a "fractal" dimension that measures roughness (more in Module 3)

Romanesco broccoli
D=2.66

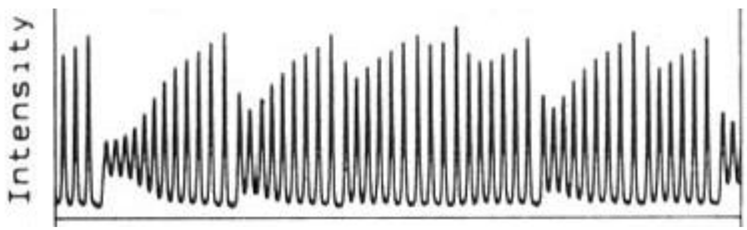
Human lung
D=2.97

Coastline of Ireland
D=1.22

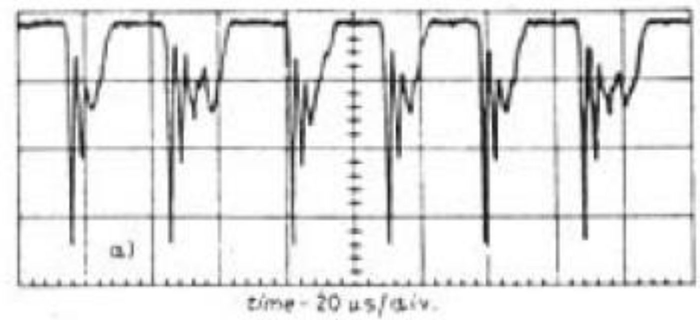
Video: http://www.ted.com/talks/benoit_mandelbrot_fractals_the_art_of_roughness#t-149180

In the 80's: can we observe chaos experimentally?

- **Optical chaos**: first observed in laser systems.



Time



… the optically pumped FIR laser system. Key work has been contributed by Vilaseca, Corbalan, and coworkers,[4] Khanin and colleagues,[5] and by Harrison, Moloney, and coworkers.[6] However, the FIR laser experiments are even …

N. B. Abraham, OPN 1989

More latter about experiments with optical chaos in our lab.

In the 90': can we control chaotic dynamics?

- **Ott**, **Grebogi** and **Yorke** (1990)

  Unstable periodic orbits can be used for control: wisely chosen **periodic kicks** can maintain the system near the desired orbit.

- **Pyragas** (1992)

  Control by using a **continuous** self-controlling **feedback** signal, whose intensity is practically zero when the system evolves close to the desired periodic orbit but increases when it drifts away.

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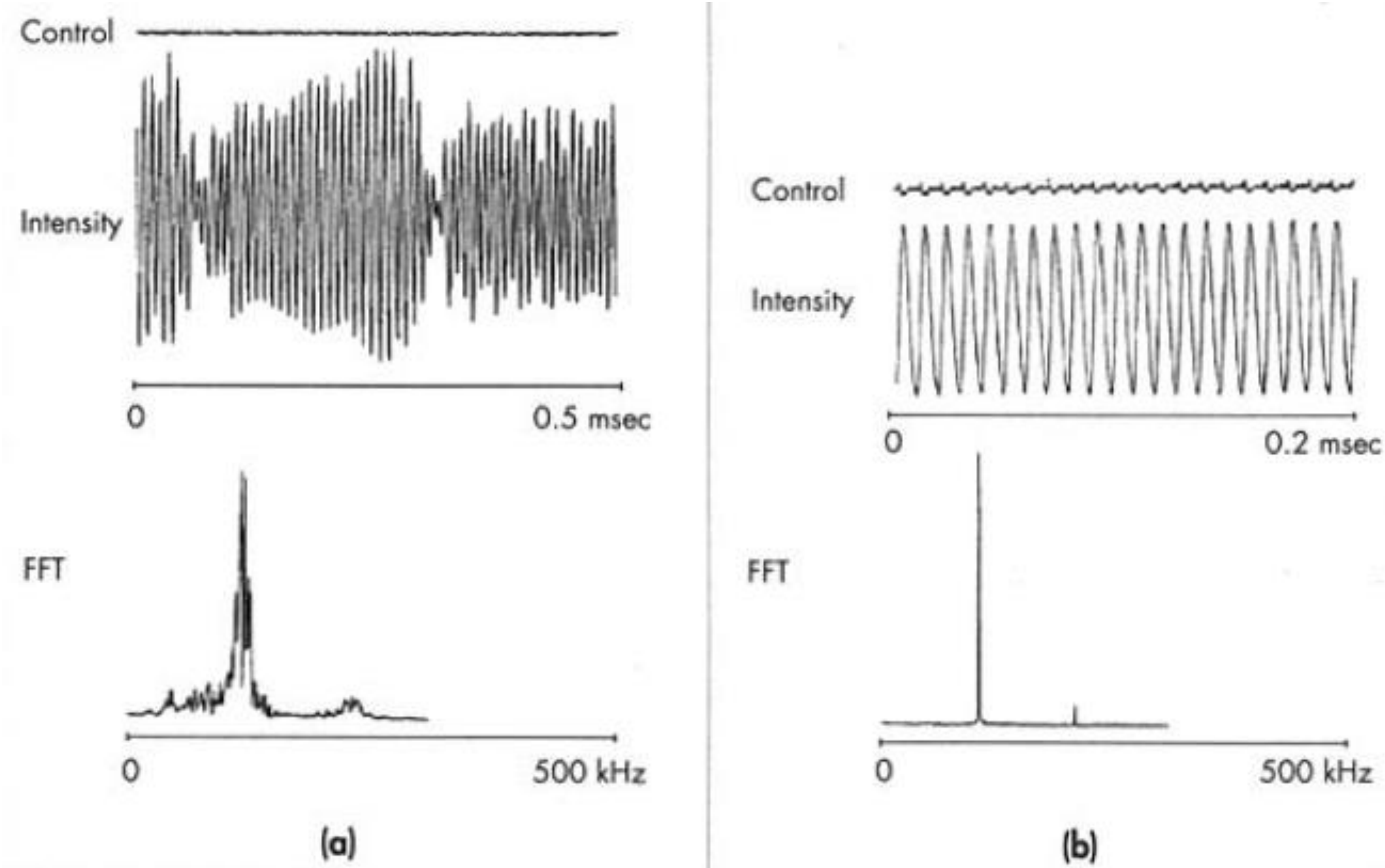
Experimental demonstration of control of optical chaos



- **Raj Roy** and others (1994)

The 1990s: synchronization of two chaotic systems

Pecora and Carroll, PRL 1990

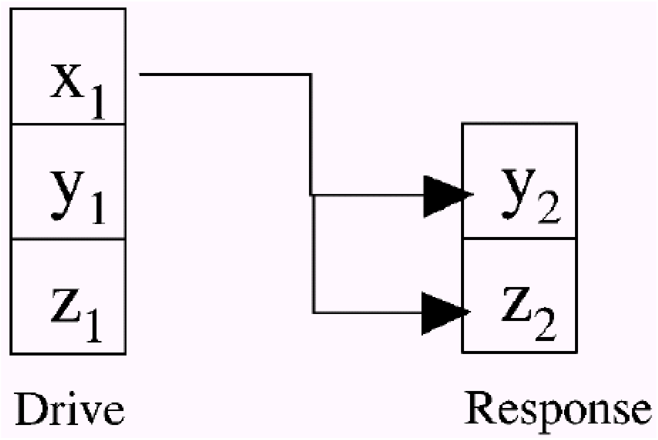
Unidirectionaly coupled Lorenz systems: the '*x*' variable of the response system is **replaced** by the '*x*' variable of the drive system.

- Drive system

$$dx_1/dt = -\sigma(y_1 - x_1)$$
$$dy_1/dt = -x_1 z_1 + r x_1 - y_1$$
$$dz_1/dt = x_1 y_1 - b z_1$$

- Response system

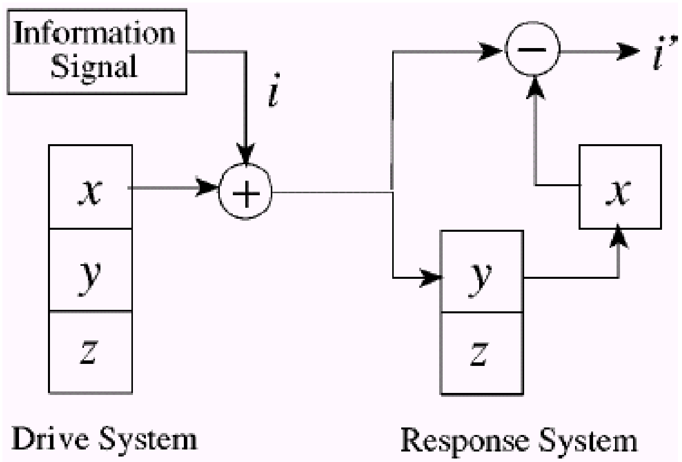
$$dy_2/dt = -x_1 y_2 + r x_1 - y_2$$
$$dz_2/dt = x_1 y_2 - b z_2$$



$$t \to \infty \quad |y_2 - y_1| \to 0, \; |z_2 - z_1| \to 0$$

Interesting but … useful?

- Transmission of secure information
  (Cuomo–Oppenheim, PRL 1993)



Is the system secure?

- In the case of low dimensional chaotic attractors, is possible to break the system and extract the message
  (G. Perez y H. Cerdeira, PRL 1995)

Different type of coupling and different types of synchronization

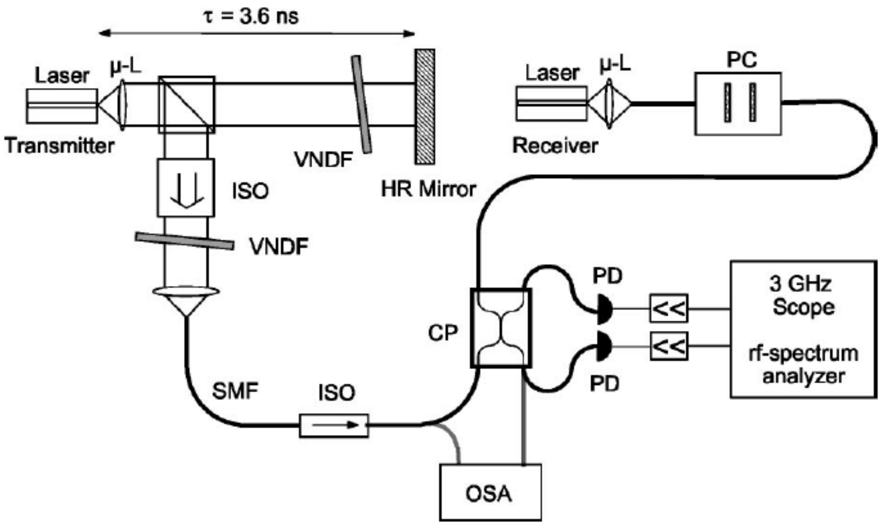
$$dx_1 / dt = F(x_1)$$

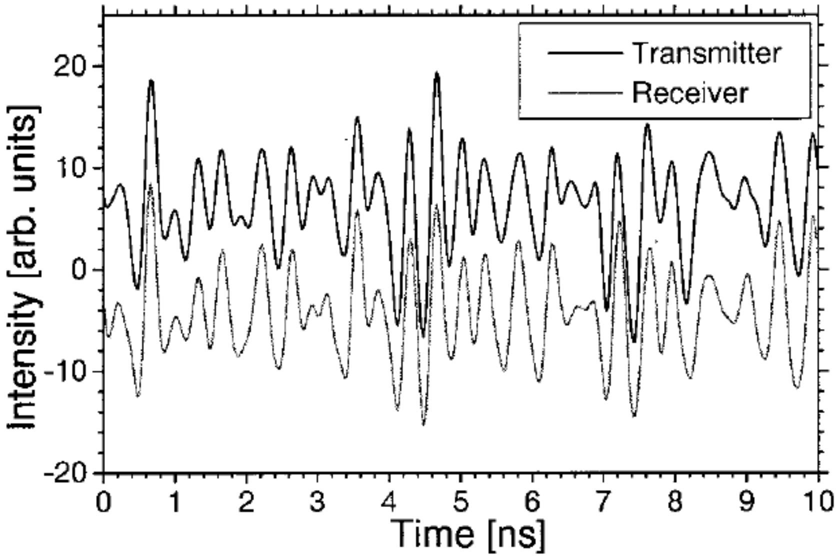
$$dx_2 / dt = F(x_2) + \alpha E(x_1 - x_2)$$

- Complete (CS): $\mathbf{x_1(t) = x_2(t)}$ (identical systems)
- Phase(PS): the phases of the oscillations are synchronized, but the amplitudes are not.
- Lag (LS): $\mathbf{x_1(t+t_o) = x_2(t)}$
- Generalized (GS): $\mathbf{x_2(t) = f(x_1(t))}$ (*f* depends on the strength of the coupling)

Experimental observation with coupled lasers

Fischer et al Phys. Rev. A 2000





Synchronization with time-delay

$$\frac{dx}{dt} = f[x(t)]$$

$$\frac{dy}{dt} = f[y(t)] + k[x(t) - y(t-\tau)]$$

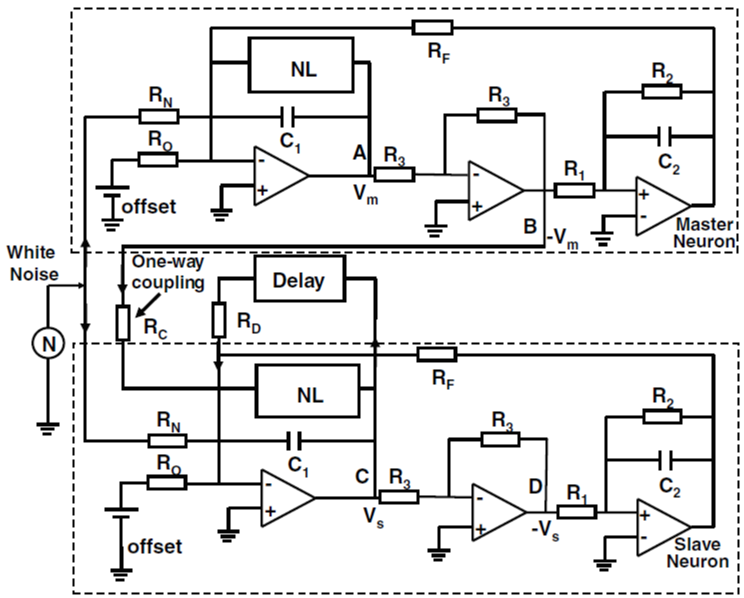
- Solution: $x(t) = y(t-\tau) \Rightarrow x(t+\tau) = y(t)$

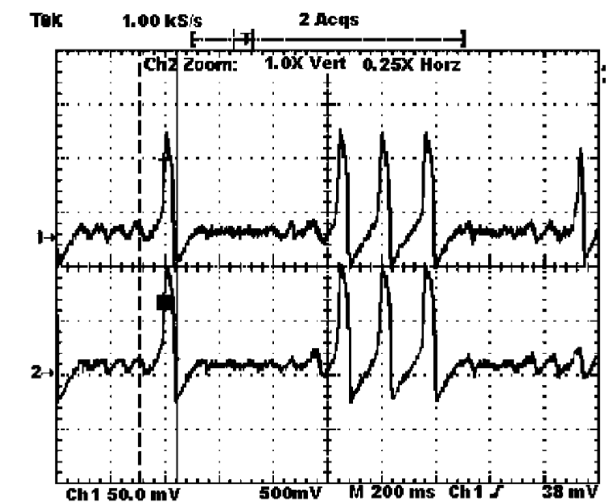
Is it possible to anticipate/predict the evolution of a chaotic system?

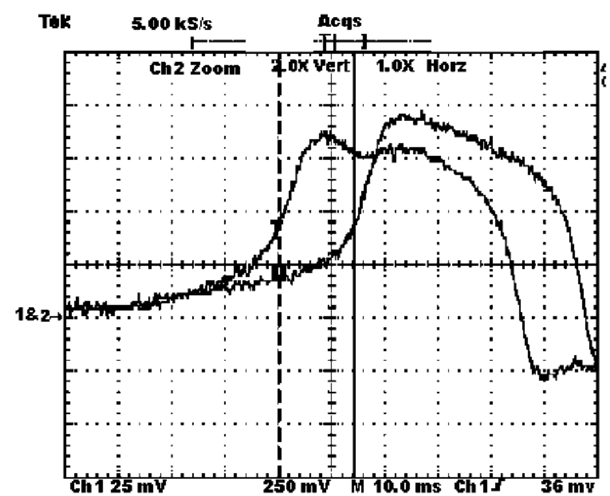
The solution is stable only for 'small' values of τ (Voss, PRL 2001)

Experimental demonstration with electronic circuits

M. Cizak et al, PRL 2003

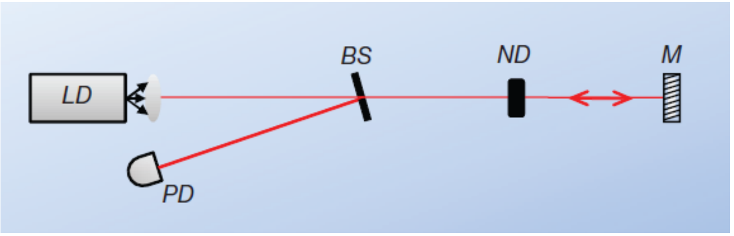


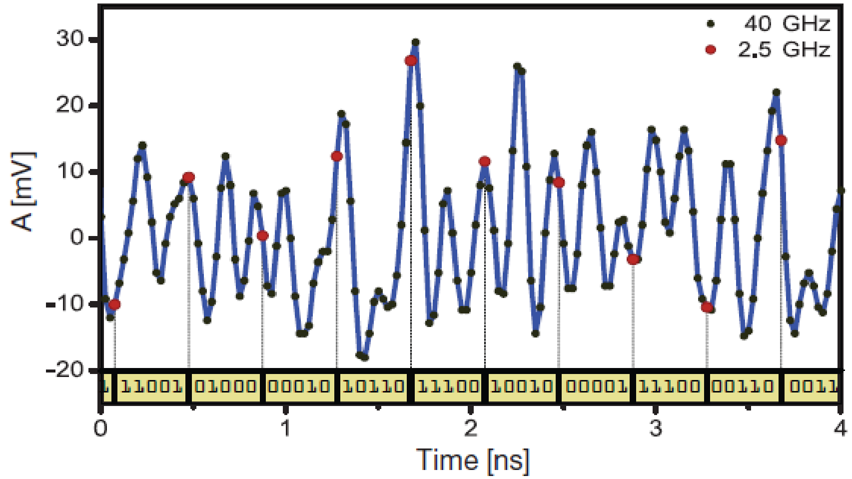




In the last decade: can we exploit / use chaotic dynamics?

An example from optical chaos



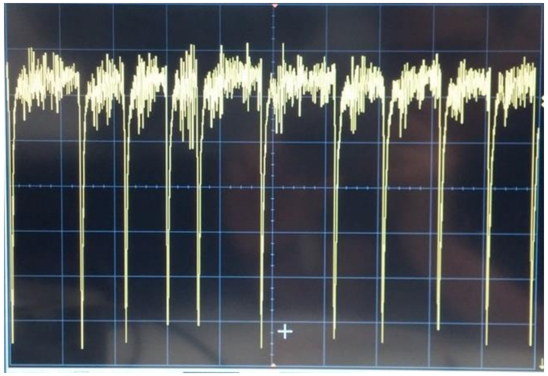


After processing the signal, **ultra-fast generation of arbitrarily long sequences of random bits**.

I. Kanter et al, Nature Photonics 2010

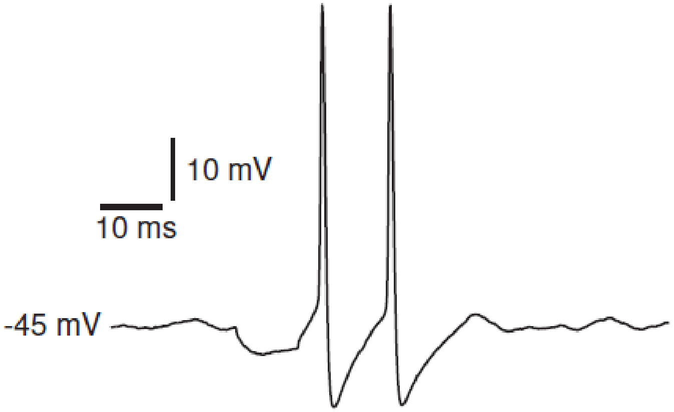
Dynamical systems in optics: Two examples from our lab

- Laser spikes: intensity vs. time

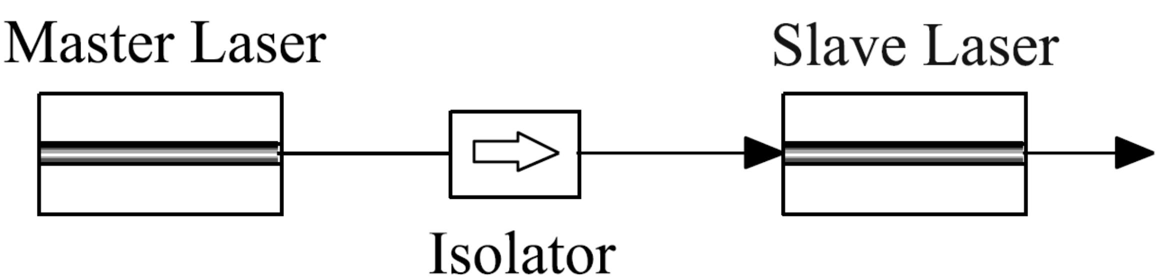


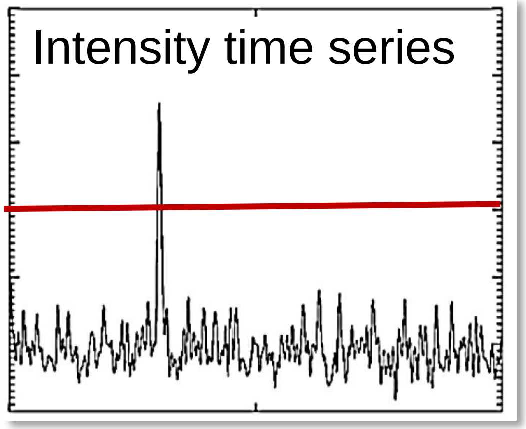
https://youtu.be/nltBQG_IIWQ

- Neuronal spikes



- Extreme pulses





Interested? TFGs available, contact us.

From dynamical systems to *complex* systems

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End of 90's - present

- Interest moves from chaotic systems to complex systems (low vs. very large number of variables).

- Networks (or graphs) of interconnected systems

- **Complexity science**: dynamics of emergent properties
  - Epidemics
  - Rumor spreading
  - Transport networks
  - Financial crises
  - Brain diseases
  - Etc.

Synchronization of a large number of coupled oscillators

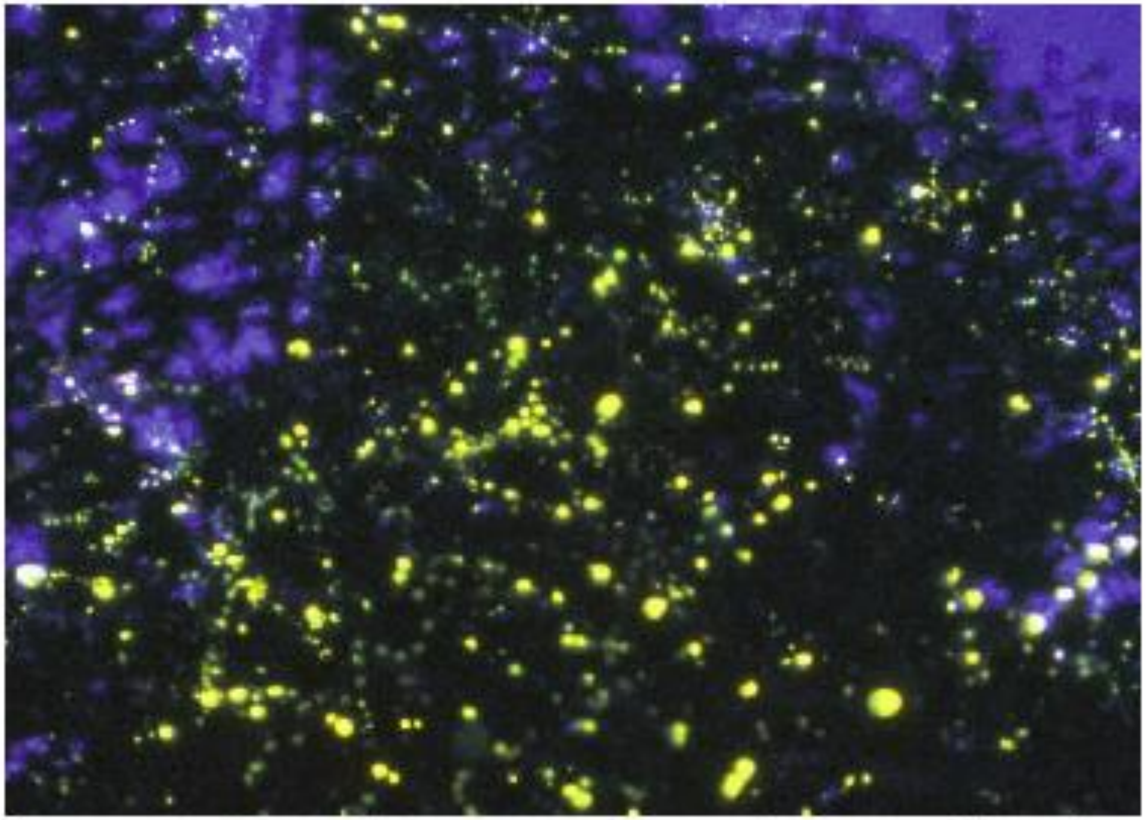


Figure 1 | Fireflies, fireflies burning bright. In the forests of the night, certain species of firefly flash in perfect synchrony — here *Pteroptyx malaccae* in a mangrove apple tree in Malaysia. Kaka *et al.*[2] and Mancoff *et al.*[3] show that the same principle can be applied to oscillators at the nanoscale.

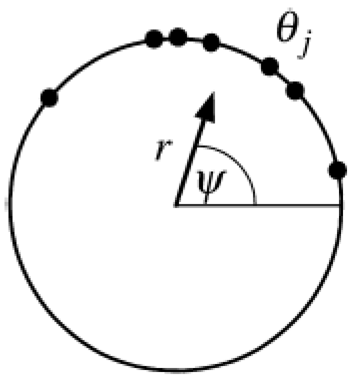
http://www.youtube.com/watch?v=DD7YDyF6dUk

Kuramoto model

(Japanese physicist, 1975)

- Model of **all-to-all** coupled **phase oscillators**.

$$\frac{d\theta_i}{dt} = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i) + \xi_i, \qquad i = 1 \ldots N$$



K = coupling strength, ξ_i = stochastic term (noise)

Describes the emergence of collective behavior (synchronization)

How to quantify synchronization?

With the order parameter: $$re^{i\psi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}$$

r =0 incoherent state (oscillators are scattered in the unit circle)

r =1 all oscillators are in phase ($\theta_i=\theta_j\ \forall$ i,j)

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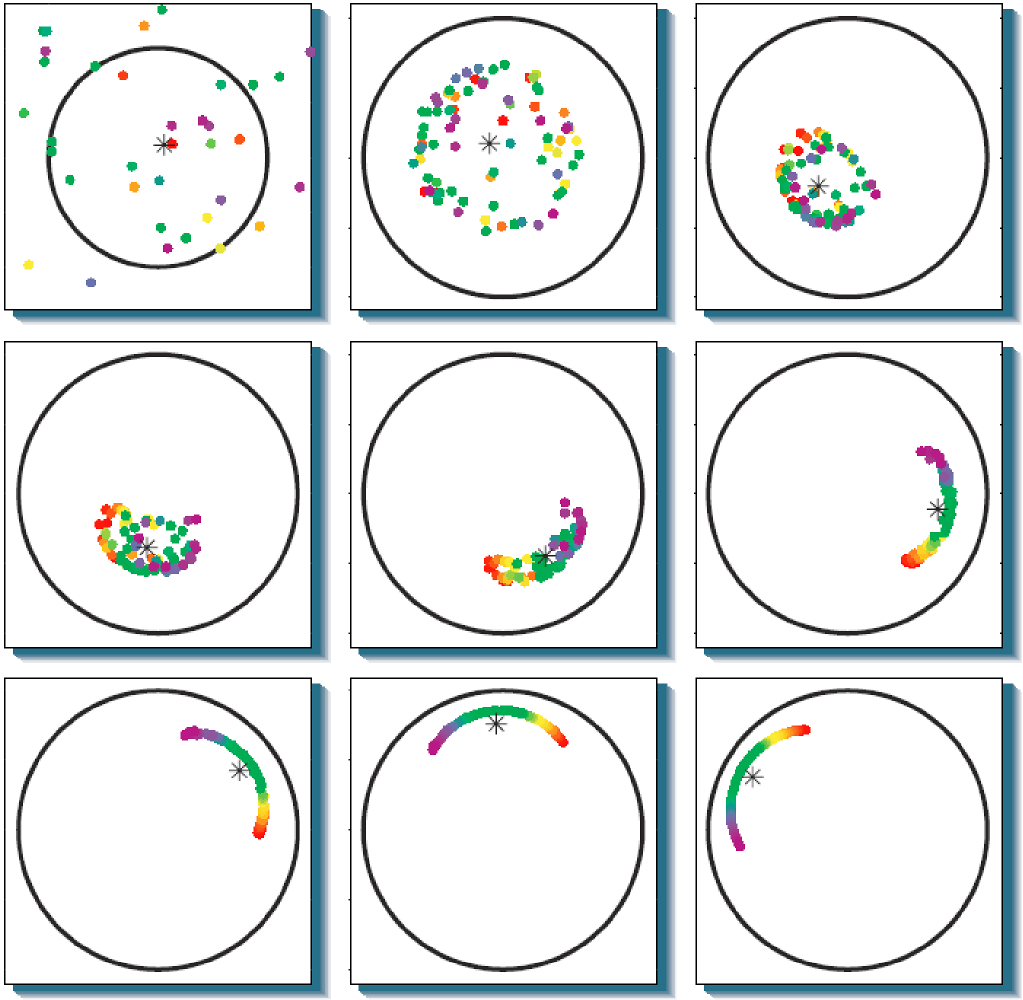
Synchronization transition as the coupling strength increases

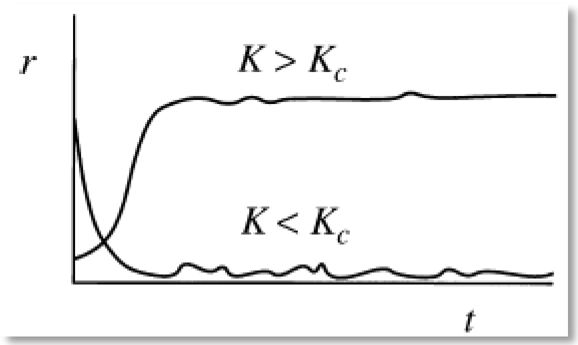


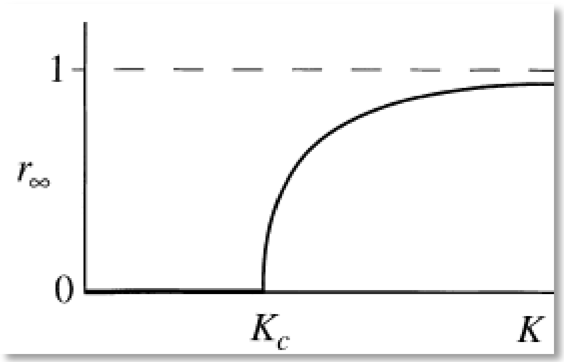
video

Strogatz
Nature 2001

Synchronization transition

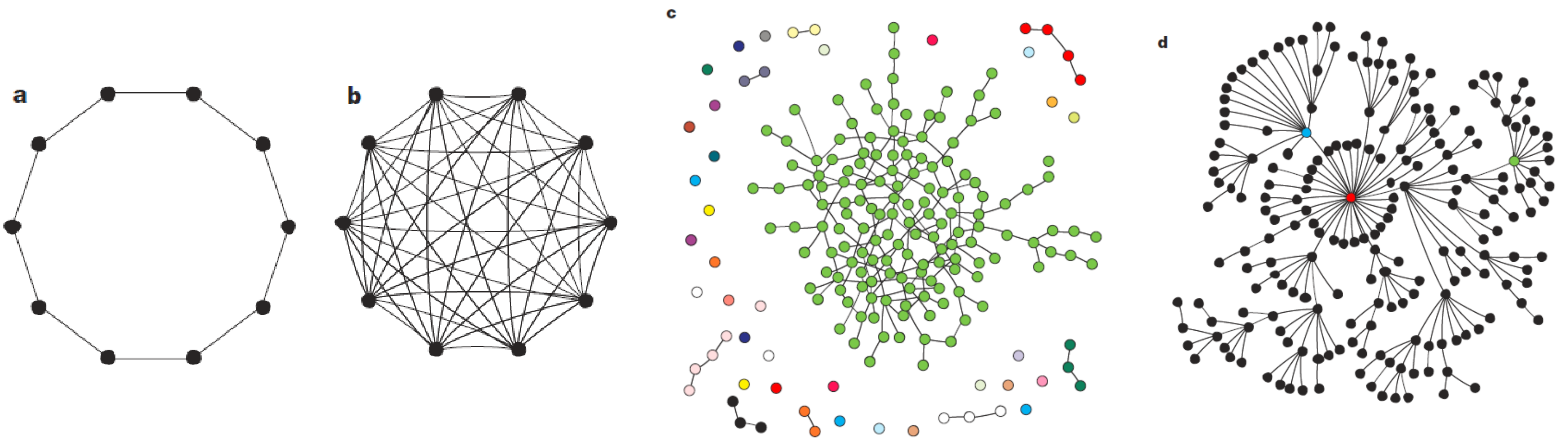
- The Kuramoto model allows for deriving exact results (**Steven Strogatz** and others, late 90').





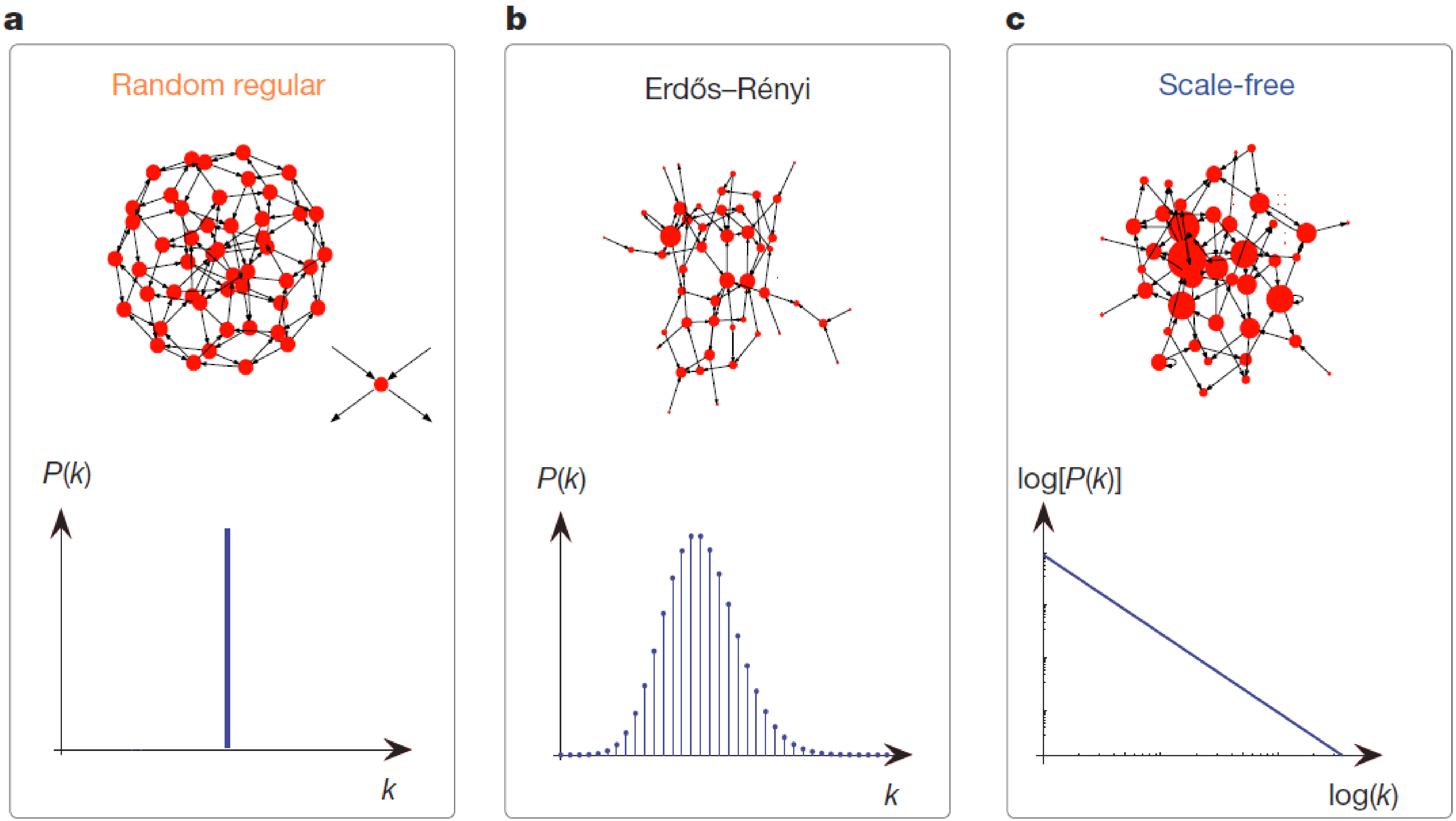
Video: https://www.ted.com/talks/steven_strogatz_on_sync

The challenge: to understand how the interplay of **structure and dynamics** (of individual units) results in emergent collective behavior.



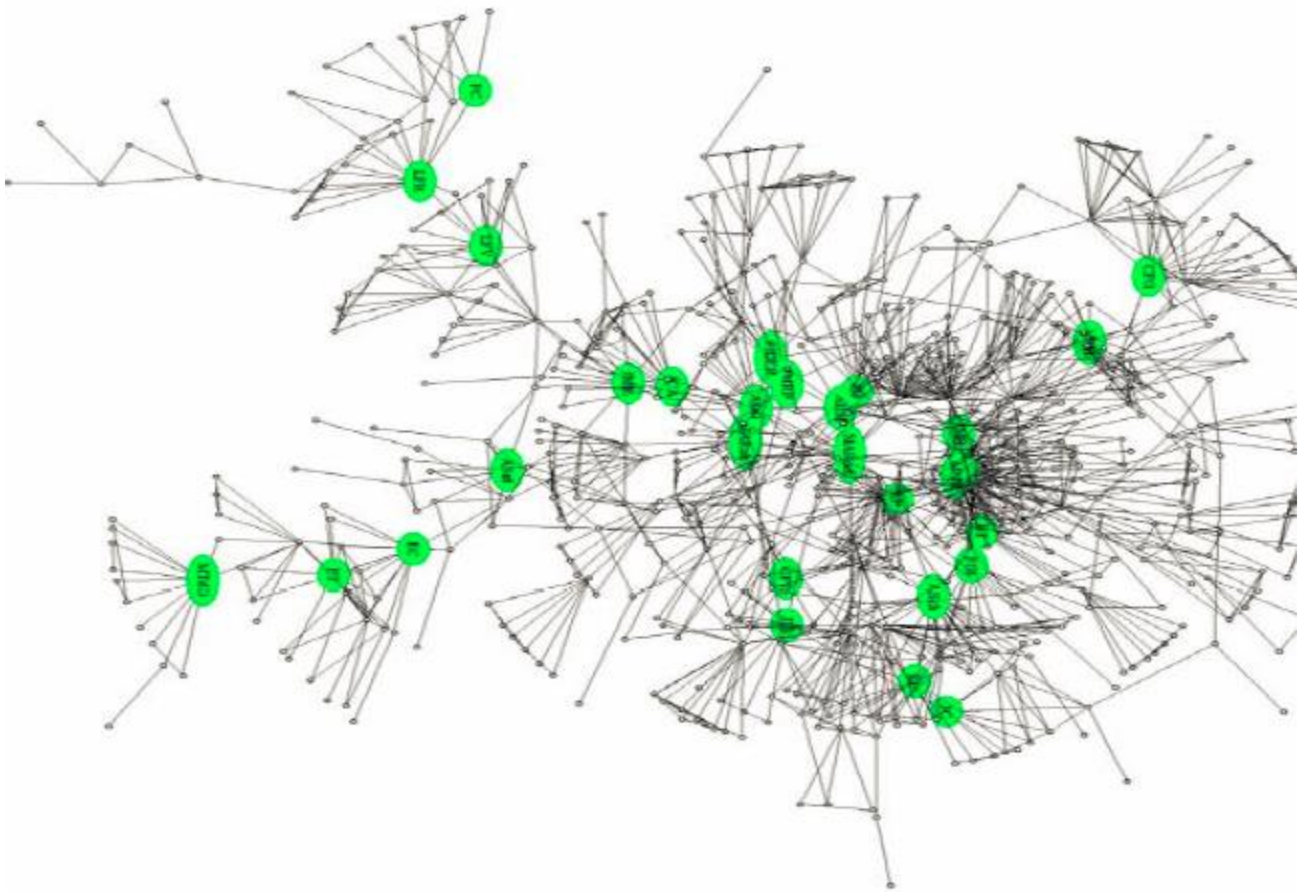
Strogatz
Nature 2001,

Main feature of a network: the degree distribution

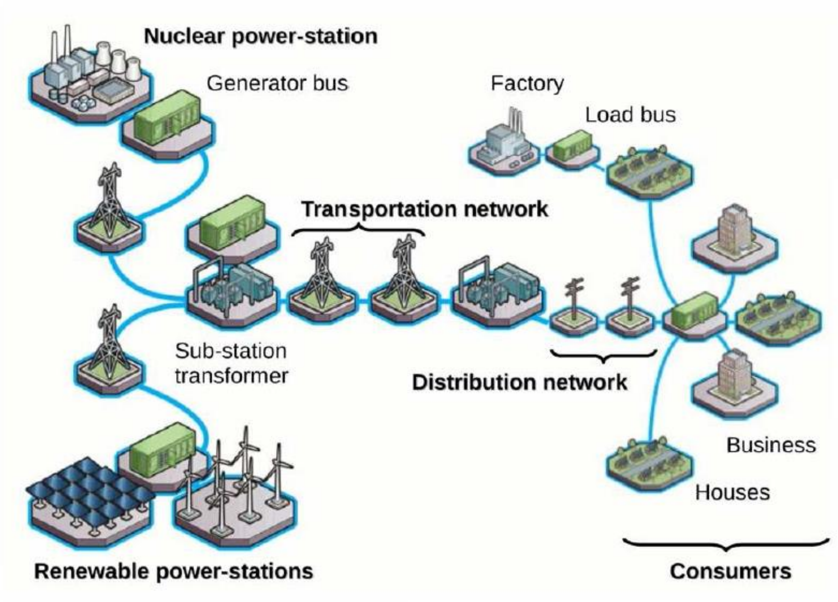


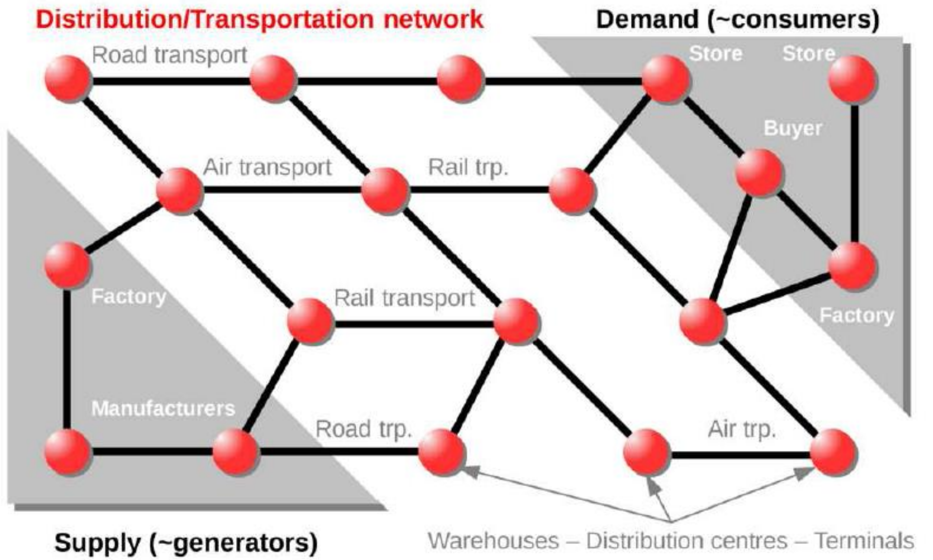
Liu et al, Nature 2011

Scientific coorperation: red FISES (A. Diaz-Guilera, UB)



Electric power grid (N. Rubido PhD Thesis 2014)





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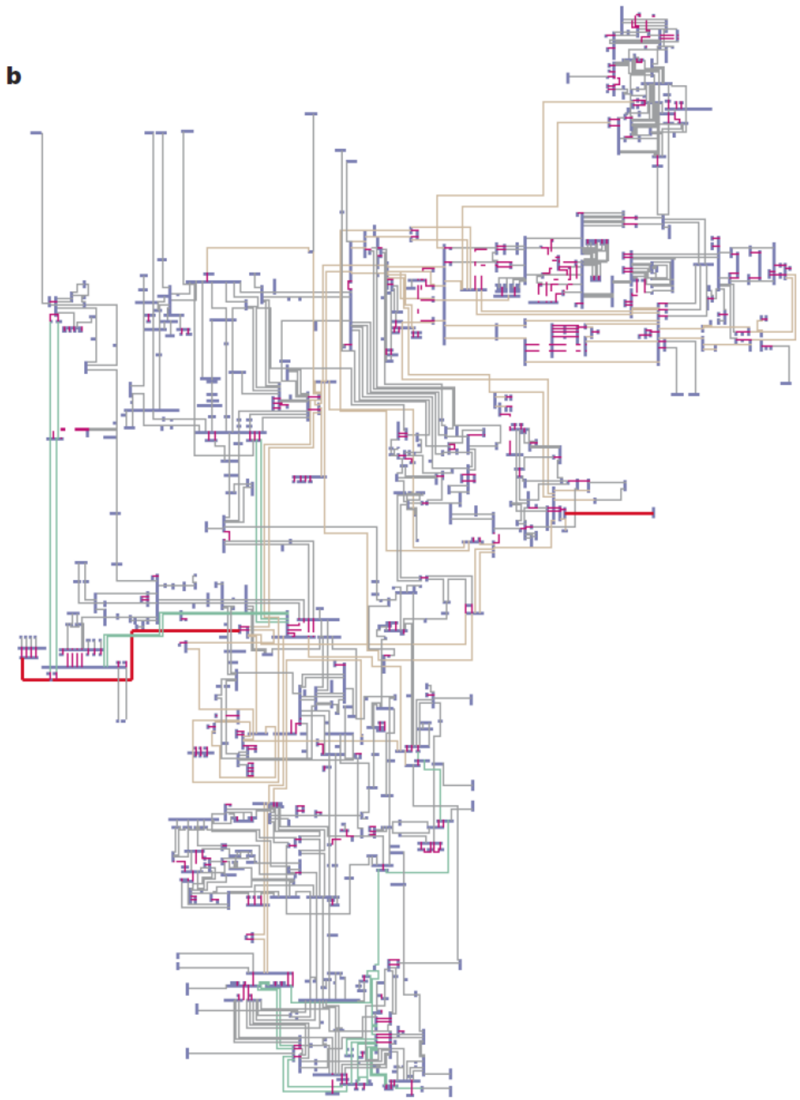
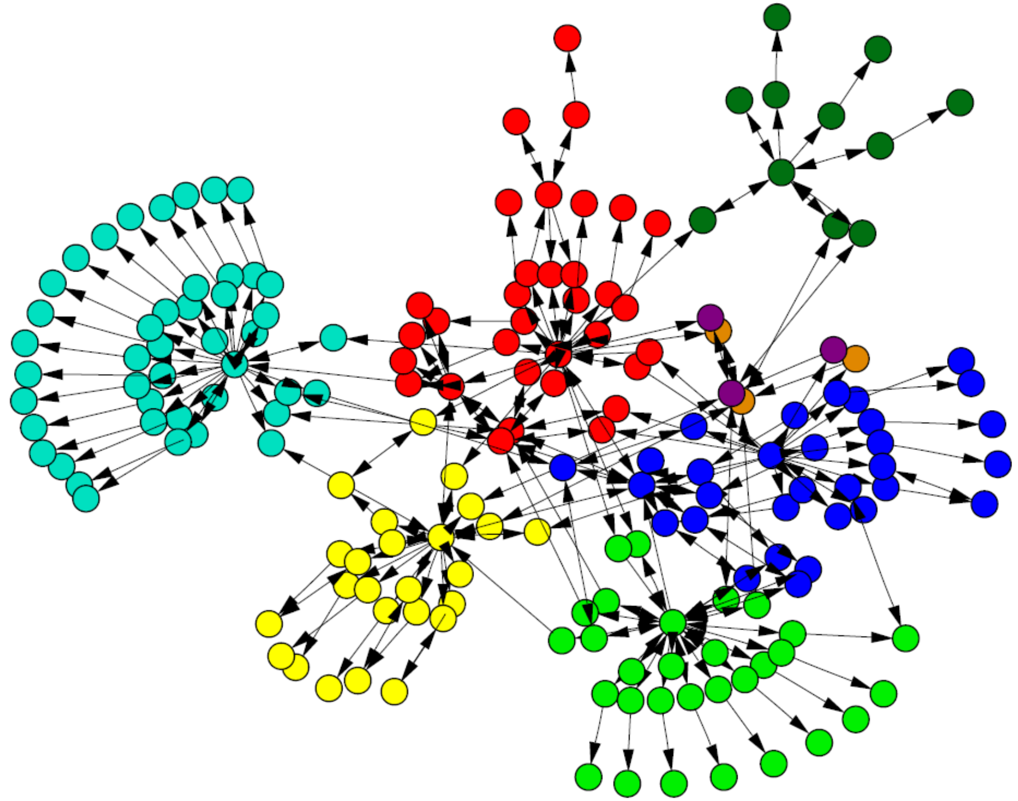
New York State electric power grid

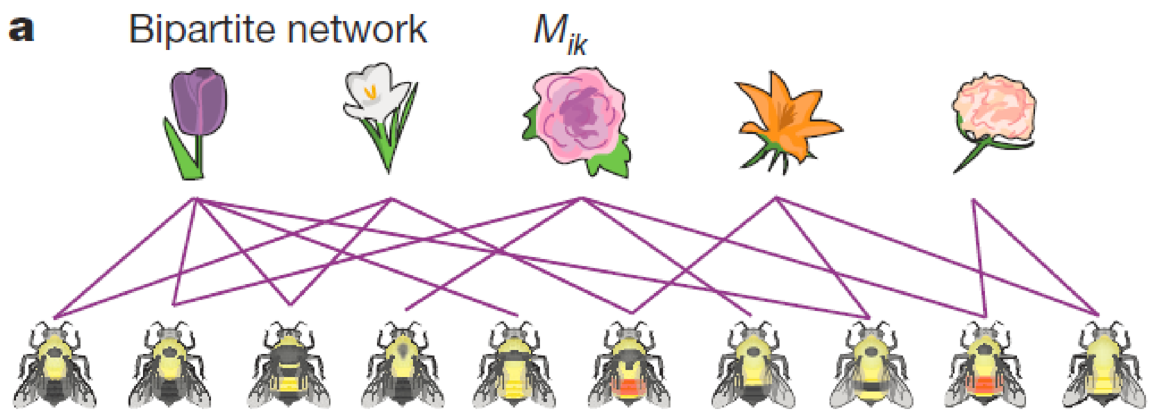


Generators and substations are shown as small blue bars. The lines connecting them are transmission lines and transformers. Line thickness and colour indicate the voltage level: red, 765 kV and 500 kV; brown, 345 kV; green, 230 kV; grey, 138 kV and below. Pink dashed lines are transformers.

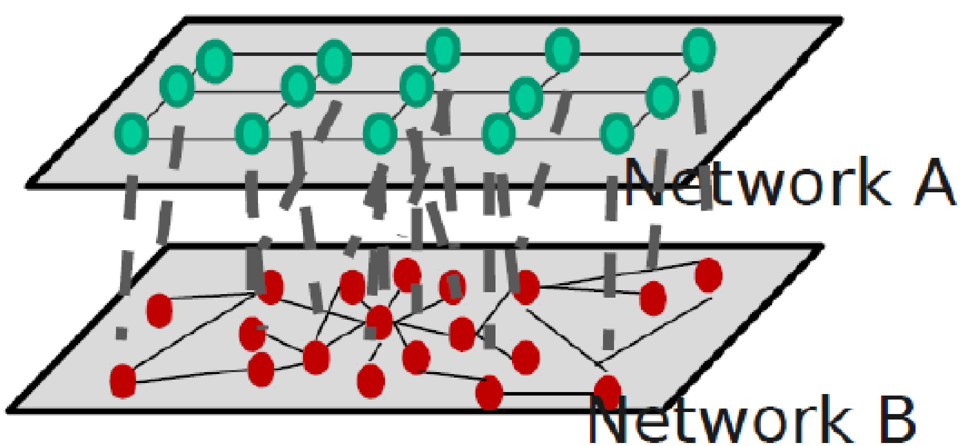
Strogatz, Nature 2001



Graph consisting of the **pages of a web site** and their mutual hyperlinks, which are directed. **Communities** are indicated by the colors (Fortunato, 2010)



Nodes represent symbiotically connected species, such as plants and pollinators (Gao et al, Nature 2016)



Interacting networks (example: friendship relations in Facebook and twitter)

Transport networks

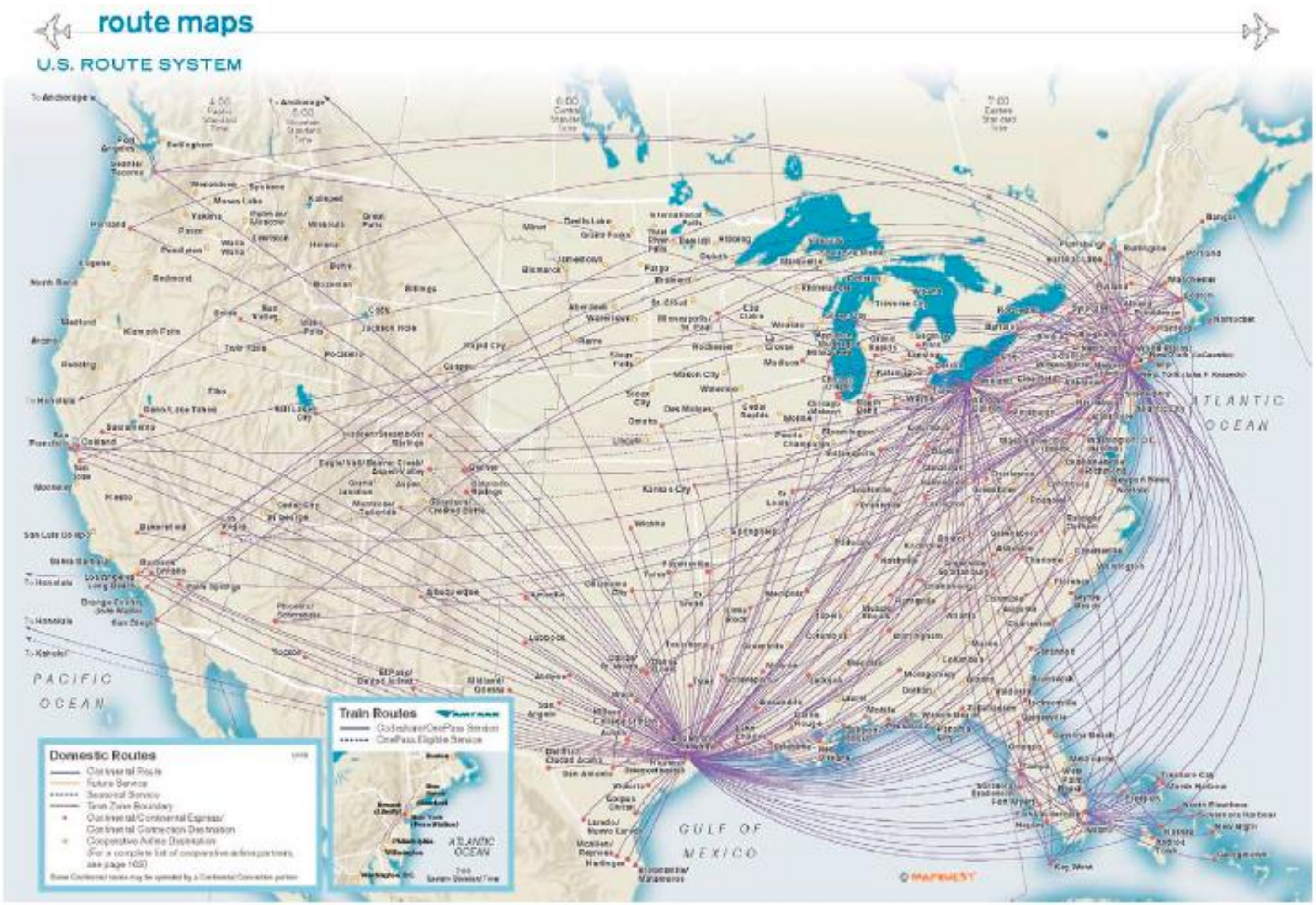
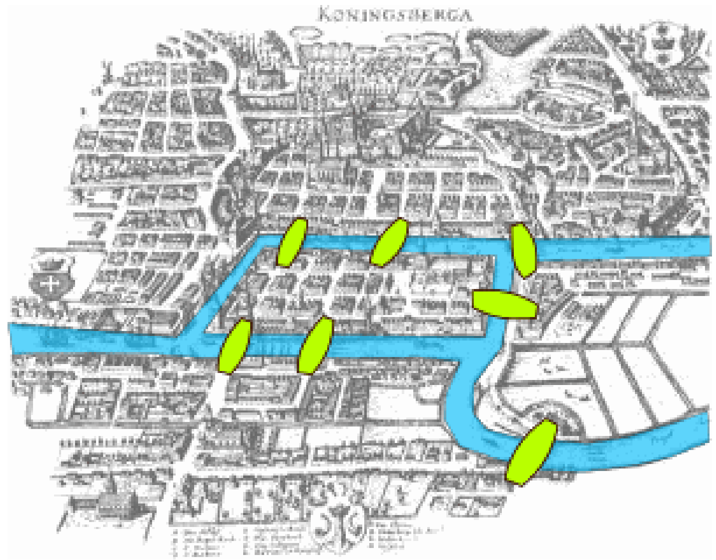
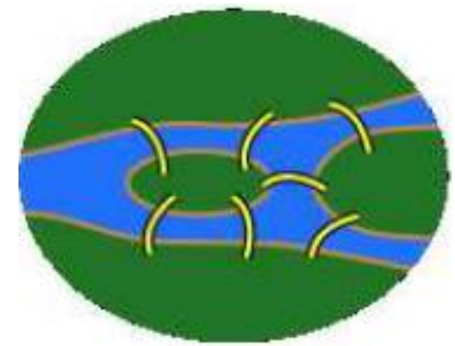


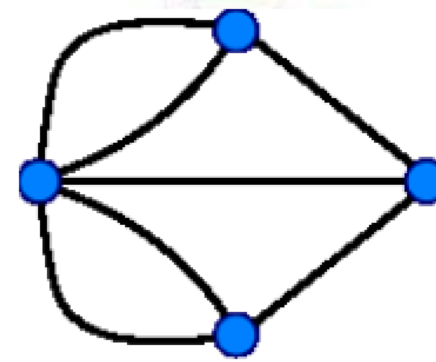
FIG. 2. Route map for Continental Airlines (courtesy of Continental Airlines).

The start of Network Theory: The Königsberg Bridge problem

The problem: to devise a walk through the city that would cross each bridge once and only once. The starting and ending points of the walk need not be the same.

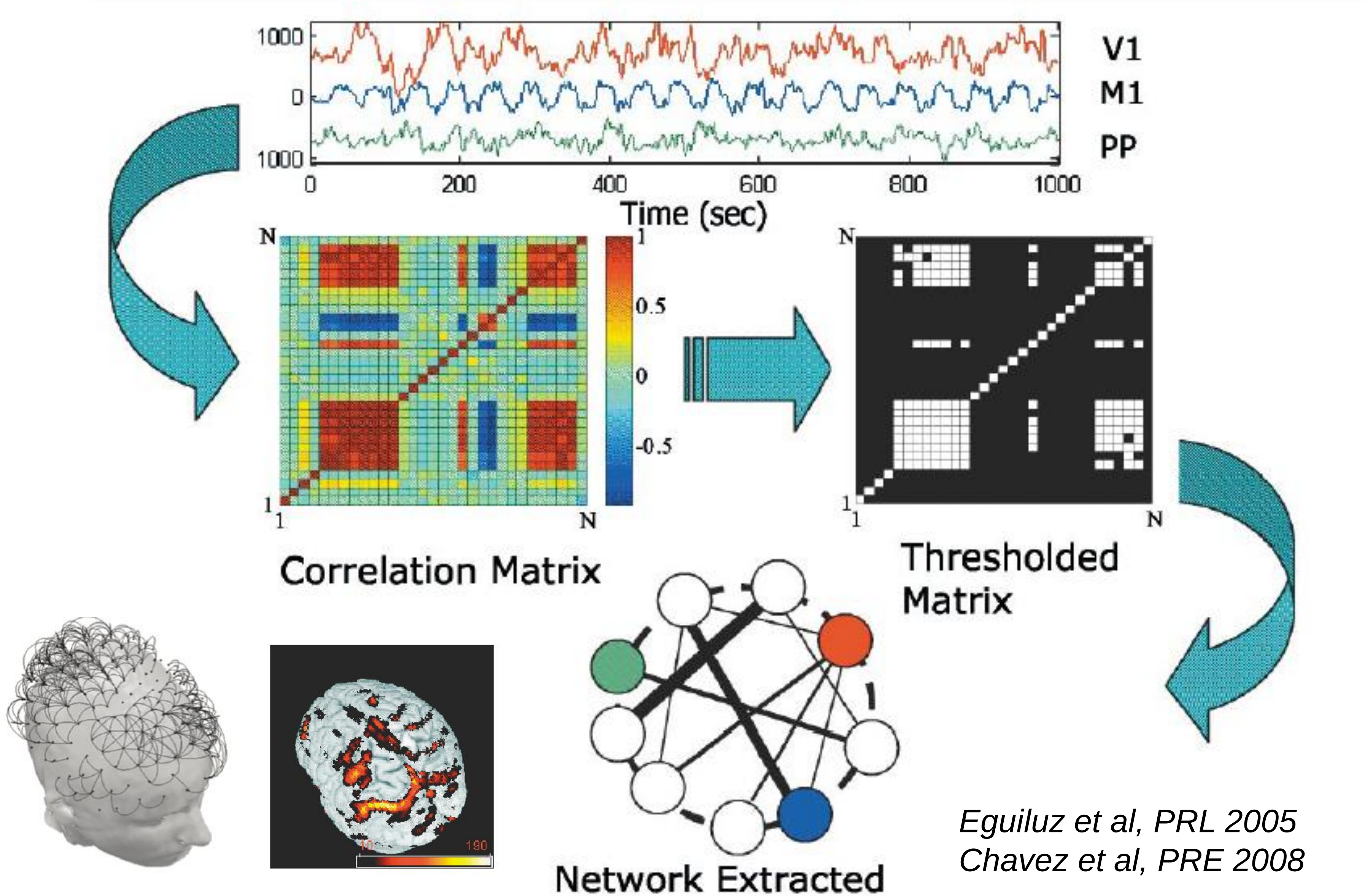






By representing the network as a **Graph** (a set of “vertices” connected by a set of “edges”) Euler (1707-1783) proved that the problem has no solution.

Brain functional network



Eguiluz et al, PRL 2005
Chavez et al, PRE 2008

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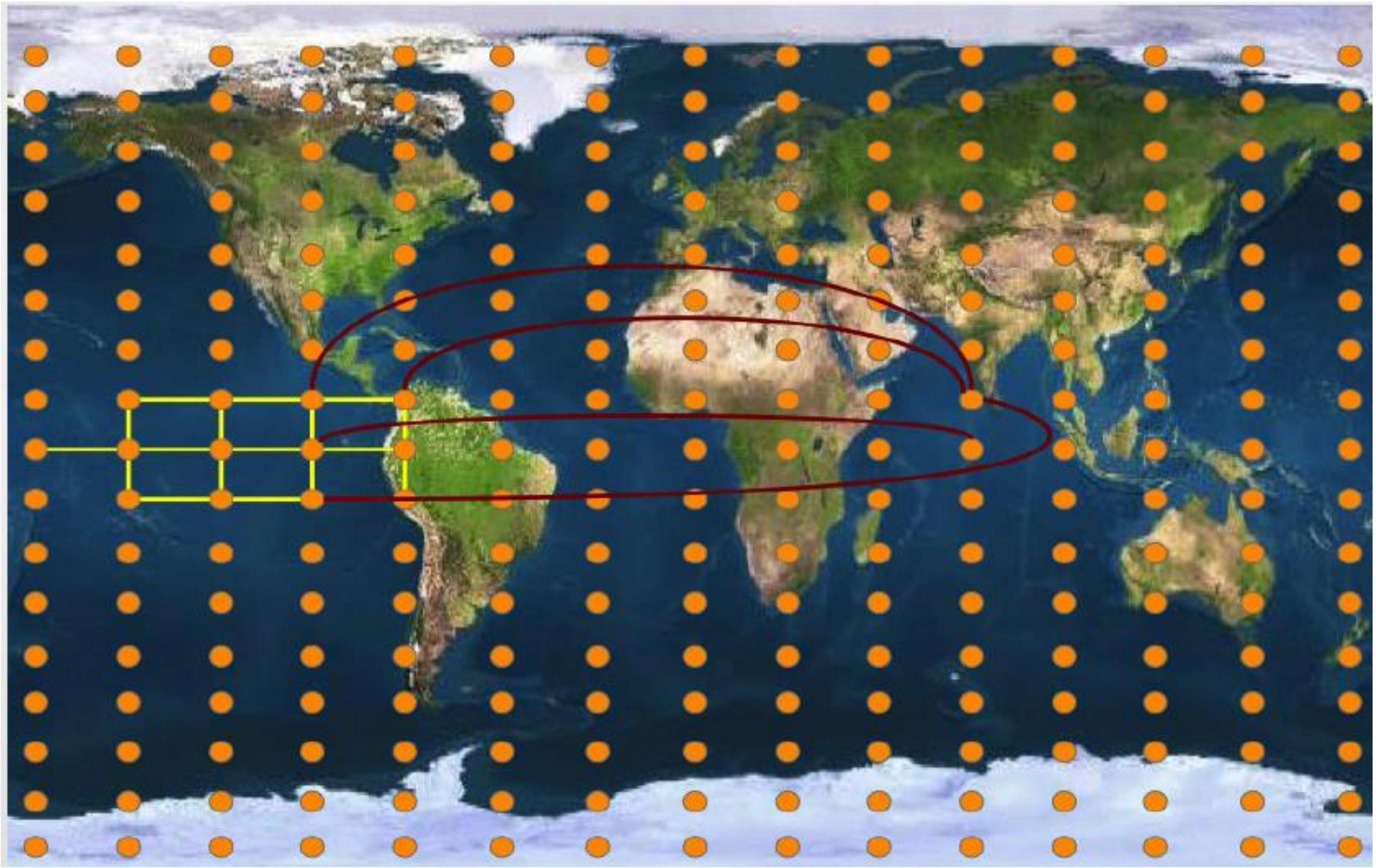
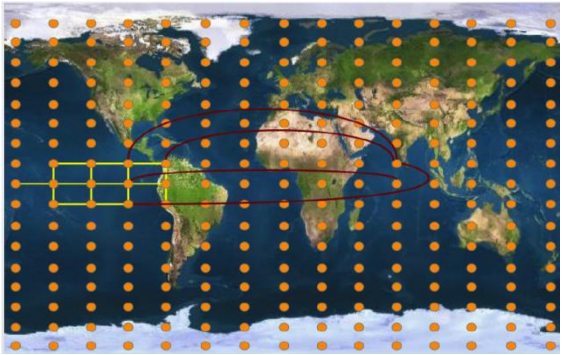
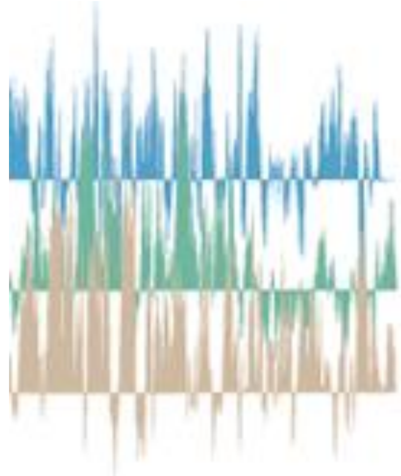
Nodes and links of the climate network

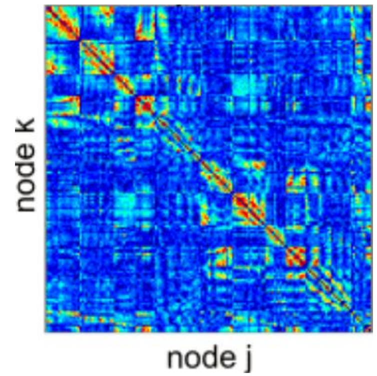


Time series of a climate variable (air temperature, wind, precipitation, etc.)

Similarity measure (correlation, mutual information, etc.)



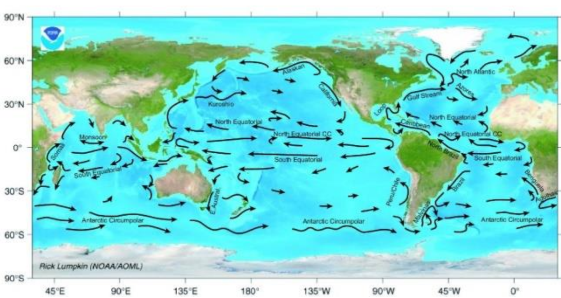




Threshold

Climate network

Winds & ocean currents



Eguiluz et al, PRL 2005
Deza et al, Chaos 2013
Donges et al, Chaos 2015

Summary and take home message

- The theory of dynamical systems allows to
  - understand the dynamics of low-dimensional systems,
  - to uncover "order within chaos",
  - universality features in the transition to chaos, and
  - provides tools for controlling chaotic behavior.

- Complexity science is aimed to understand the emerging phenomena in large sets of interacting systems.

- Both, dynamical systems and complexity science have multiple applications, and involve the work of mathematicians, physicists, biologists, computer scientists, engineers, etc.

Outline

- Introduction to dynamical systems
- **Introduction to flows on the line**
- Solving equations with computer
- Fixed points and linear stability
- Feedback control: delay differential equations

Types of dynamical systems

- **Continuous time**: differential equations

  - Ordinary differential equations (ODEs).
    Example: damped oscillator

$$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=0$$

  - Partial differential equations (PDEs).
    Example: heat equation

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$$

- **Discrete time**: difference equations or "iterated maps". Example: the logistic map

$$x(i+1)=\mathbf{r}\ x(i)[1-x(i)]$$

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ODEs can be written as first-order differential equations

$$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=0 \quad \Rightarrow \quad \begin{aligned}\dot{x}_1 &= f_1(x_1,\ldots,x_n)\\ &\vdots\\ \dot{x}_n &= f_n(x_1,\ldots,x_n)\end{aligned} \qquad \boxed{\dot{x}=f(x)}$$

- First example: harmonic oscillator $m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=0$

$$x_1 = x \text{ and } x_2 = \dot{x}$$

$$\dot{x}_2 = \ddot{x} = -\tfrac{b}{m}\dot{x} - \tfrac{k}{m}x = -\tfrac{b}{m}x_2 - \tfrac{k}{m}x_1 \quad \Rightarrow \quad \boxed{\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\tfrac{b}{m}x_2 - \tfrac{k}{m}x_1\end{aligned}}$$

- Second example: pendulum

$$\ddot{x}+\tfrac{g}{L}\sin x = 0 \quad \Rightarrow \quad \boxed{\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\tfrac{g}{L}\sin x_1\end{aligned}}$$

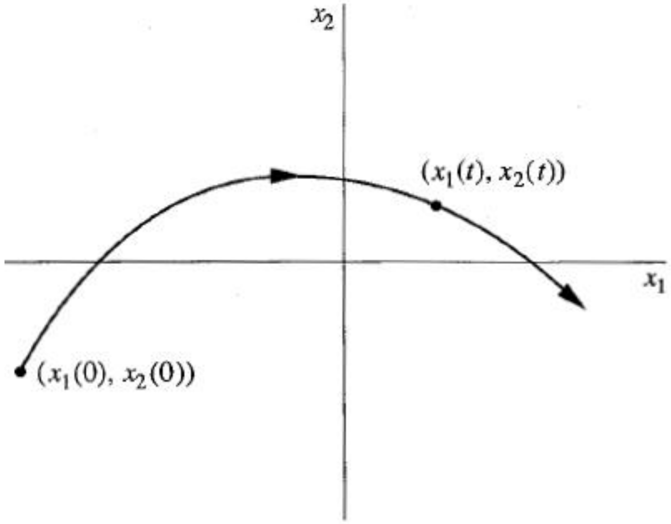
Trajectory in the phase space

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{b}{m} x_2 - \frac{k}{m} x_1$$

- Given the initial conditions, $x_1(0)$ and $x_2(0)$, we predict the evolution of the system by solving the equations: $x_1(t)$ and $x_2(t)$.
- $x_1(t)$ and $x_2(t)$ are solutions of the equations.

- The evolution of the system can be represented as a <u>trajectory</u> in the <u>phase space</u>.

$\Rightarrow$ two-dimensional (2D) dynamical system.



Key argument (Poincare): find out how the trajectories look like, without solving the equations explicitly.

Classification of dynamical systems described by ODEs (I/II)

$$\boxed{\dot{x} = f(x) + \xi(t)}$$

- *f(x)* linear: in the function *f*, *x* appears to first order only (no x^2, x_1x_2, *sin(x)* etc.). Then, the behavior can be understood from the sum of its parts.
- *f(x)* nonlinear: superposition principle fails!
- Example of linear system: harmonic oscillator

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0 \quad \Rightarrow \quad \boxed{\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\tfrac{b}{m}x_2 - \tfrac{k}{m}x_1 \end{aligned}}$$

In the right-hand-side x_1 and x_2 appear to first power (no products etc.)

- Example of nonlinear system: pendulum

$$\ddot{x} + \tfrac{g}{L}\sin x = 0 \quad \Rightarrow \quad \boxed{\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\tfrac{g}{L}\sin x_1 \end{aligned}}$$

Classification of dynamical systems described by ODEs (II/II)

$$\boxed{\dot{x} = f(x) + \xi(t)}$$

- ξ=0: deterministic.
- $\xi \neq$0: stochastic (real life) –simplest case: additive noise.

- *x*: vector with few variables (*n*<4): low dimensional.
- *x*: vector with many variables: high dimensional.

- *f* does not depend on time: autonomous system.
- *f* depends on time: non-autonomous system.

Example of non-autonomous system: a forced oscillator

$$m\ddot{x} + b\dot{x} + kx = F\cos t$$

- Can also be written as first-order ODE

$$x_1 = x \text{ and } x_2 = \dot{x} \qquad x_3 = t \qquad \dot{x}_3 = 1 \quad \Rightarrow \quad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \tfrac{1}{m}\left(-kx_1 - bx_2 + F\cos x_3\right) \\ \dot{x}_3 &= 1 \end{aligned}$$

- 3D system: to predict the future evolution we need to know the present state (*t, x, dx/dt*).

So…what is a "flow on the line"?

- A one-dimensional autonomous dynamical system described by a first-order ordinary differential equation

$$\dot{x} = f(x)$$

- x $\in \Re$
- *f* does not depend on time

Summarizing

Number of variables

	N=1	N=2	N=3	N>>1	N=∞ (PDEs DDEs)
Linear	RC circuit	Harmonic oscillator	$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = A\mathbf{x}(t) + B\mathbf{u}(t)$ $\mathbf{x}(t) = (x_1(t), \ \ldots, \ x_N(t))$		• Heat equation, • Maxwell equations • Schrodinger equation
Nonlinear	Logistic population grow	Pendulum	• Forced oscillator • Lorentz model	• Kuramoto phase oscillators	• Navier-Stokes (turbulence)

"flow on the line"

PDEs=partial differential eqs.
DDEs=delay differential eqs.

Outline

- Introduction to dynamical systems
- Introduction to flows on the line
- **Solving equations with computer**
- Fixed points and linear stability
- Feedback control: delay differential equations

Numerical integration

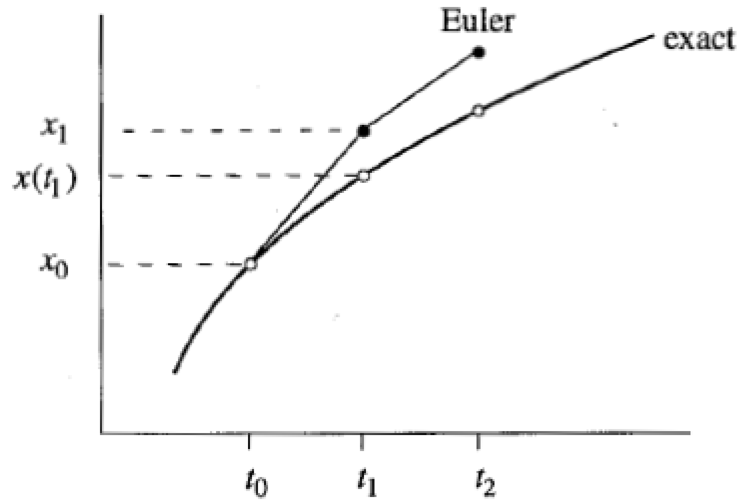
$$\dot{x} = f(x)$$

- Euler method

$$x(t_0 + \Delta t) \approx x_1 = x_0 + f(x_0)\Delta t$$

$$x_{n+1} = x_n + f(x_n)\Delta t$$

$$t_n = t_0 + n\Delta t$$



- Euler second order

$$\tilde{x}_{n+1} = x_n + f(x_n)\Delta t$$

$$x_{n+1} = x_n + \tfrac{1}{2}\left[f(x_n) + f(\tilde{x}_{n+1})\right]\Delta t$$

- Fourth order (Runge-Kutta 1905)

$$k_1 = f(x_n)\Delta t$$

$$k_2 = f(x_n + \tfrac{1}{2}k_1)\Delta t$$

$$k_3 = f(x_n + \tfrac{1}{2}k_2)\Delta t$$

$$k_4 = f(x_n + k_3)\Delta t\,.$$

$$x_{n+1} = x_n + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

- Problem if Δt is too small: round-off errors (computers have finite accuracy).

Table 12.1. *MATLAB's ODE solvers.*

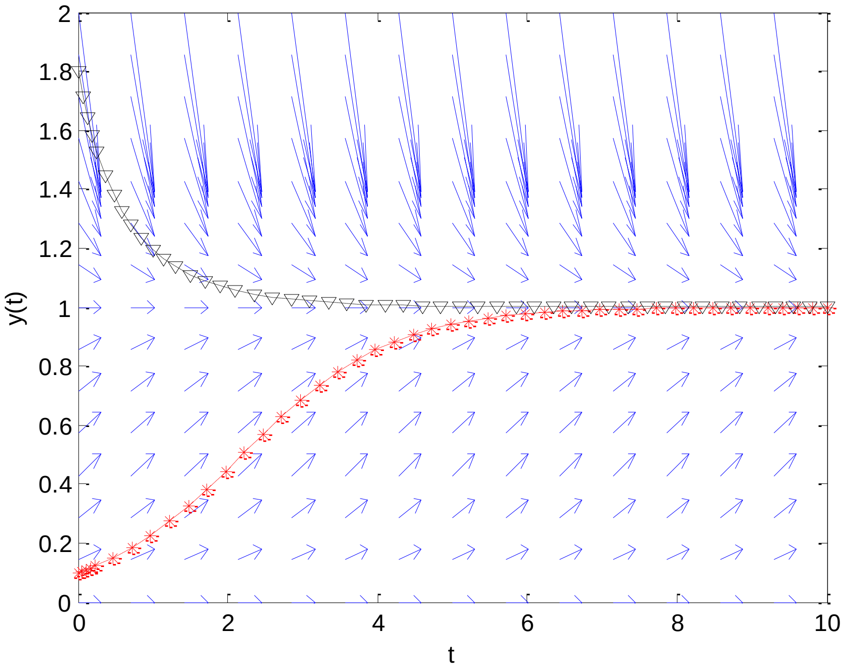
Solver	Problem type	Type of algorithm
`ode45`	Nonstiff	Explicit Runge–Kutta pair, orders 4 and 5
`ode23`	Nonstiff	Explicit Runge–Kutta pair, orders 2 and 3
`ode113`	Nonstiff	Explicit linear multistep, orders 1 to 13
`ode15s`	Stiff	Implicit linear multistep, orders 1 to 5
`ode23s`	Stiff	Modified Rosenbrock pair (one-step), orders 2 and 3
`ode23t`	Mildly stiff	Trapezoidal rule (implicit), orders 2 and 3
`ode23tb`	Stiff	Implicit Runge–Kutta type algorithm, orders 2 and 3

Example 1

- **quiver**(x,y,u,v,*scale*): plots arrows with components (u,v) at the location (x,y).

- The length of the arrows is *scale* times the norm of the (u,v) vector.

$$\dot{y} = y(1 - y)$$



To plot the blue arrows:

```
%vector_field.m
n=15;
tpts = linspace(0,10,n);
ypts = linspace(0,2,n);
[t,y] = meshgrid(tpts,ypts);
pt = ones(size(y));
py =  y.*(1-y);
quiver(t,y,pt,py,1);
xlim([0 10]), ylim([0 2])
```

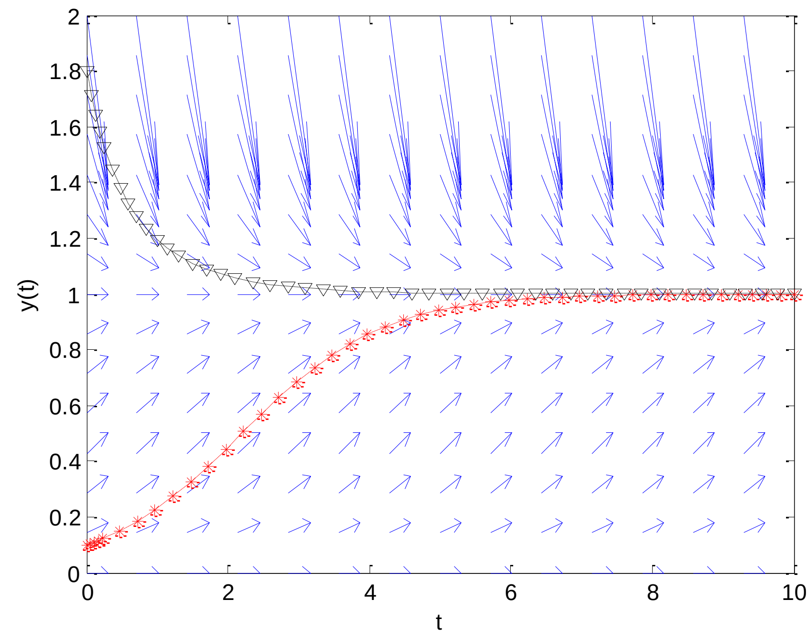
Numerical solution

$$\dot{y} = y(1-y) \qquad y(0) = 0.1$$

To plot the solution (in red):

```
tspan = [0 10];
yzero = 0.1;
[t, y] =ode45(@myf,tspan,yzero);
plot(t,y,'r*--'); xlabel t; ylabel y(t)
```

```
function yprime = myf(t,y)
yprime = y.*(1-y);
```



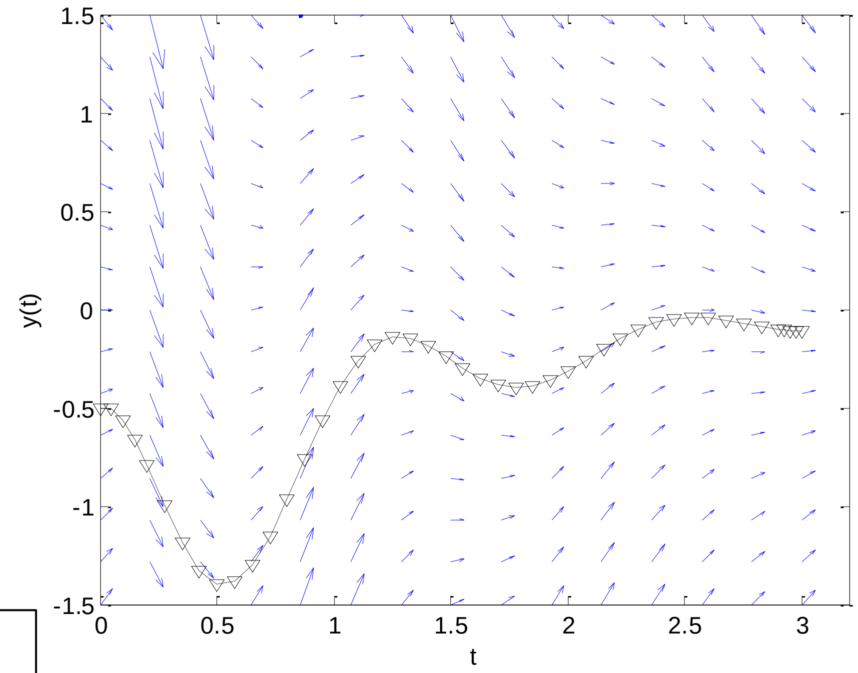
The solution is always tangent to the arrows

Remember: HOLD to plot together the blue arrows & the trajectory.

Example 2

$$\dot{y} = -y - 5e^{-t}\sin 5t \qquad y(0) = -0.5$$

```
n=15;
tpts = linspace(0,3,n);
ypts = linspace(-1.5,1.5,n);
[t,y] = meshgrid(tpts,ypts);
pt = ones(size(y));
py = -y-5*exp(-t).*sin(5*t);
quiver(t,y,pt,py,1);
xlim ([0 3.2]), ylim([-1.5 1.5])
```




```
tspan = [0 3];
yzero = -0.5;
[t, y] = ode45(@myf,tspan,yzero);
plot(t,y,'kv--'); xlabel t; ylabel y(t)
```

```
function yprime = myf(t,y)
yprime = -y -5*exp(-t)*sin(5*t);
```

General form of a call to Ode45

```
[t,y] = ode45(@fun,tspan,yzero,options,p1,p2,...);
```

The optional trailing arguments `p1`, `p2`, ... represent problem parameters that, if provided, are passed on to the function `fun`. The optional argument `options` is a structure that controls many features of the solver and can be set via the `odeset` function. In our next example we create a structure `options` by the assignment

```
options = odeset('AbsTol',1e-7,'RelTol',1e-4);
```

Passing this structure as an input argument to `ode45` causes the absolute and relative error tolerances to be set to 10^{-7} and 10^{-4}, respectively. (The default values are 10^{-6} and 10^{-3}; see `help odeset` for the precise meaning of the tolerances.) These

Class and homework

2.8.3 (Calibrating the Euler method) The goal of this problem is to test the Euler method on the initial value problem $\dot{x} = -x$, $x(0) = 1$.

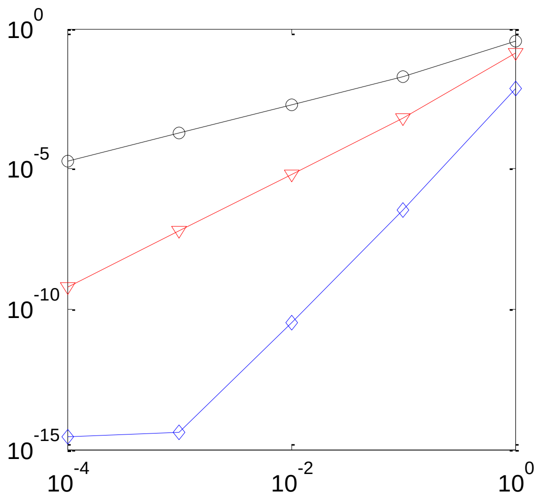
a) Solve the problem analytically. What is the exact value of $x(1)$?

b) Using the Euler method with step size $\Delta t = 1$, estimate $x(1)$ numerically—call the result $\hat{x}(1)$. Then repeat, using $\Delta t = 10^{-n}$, for $n = 1, 2, 3, 4$.

c) Plot the error $E = |\hat{x}(1) - x(1)|$ as a function of Δt. Then plot $\ln E$ vs. $\ln t$. Explain the results.

2.8.4 Redo Exercise 2.8.3, using the improved Euler method.

2.8.5 Redo Exercise 2.8.3, using the Runge–Kutta method.



Outline

- Introduction to dynamical systems
- Introduction to flows on the line
- Solving equations with computer
- **Fixed points and linear stability**
- Feedback control: delay differential equations

Example

$$\dot{x} = \sin x$$

Analytical Solution: $t = \ln \left| \dfrac{\csc x_0 + \cot x_0}{\csc x + \cot x} \right|$

- Starting from $x_0 = \pi/4$, what is the long-term behavior (what happens when $t \to \infty$?)

- And for any arbitrary condition x_o?

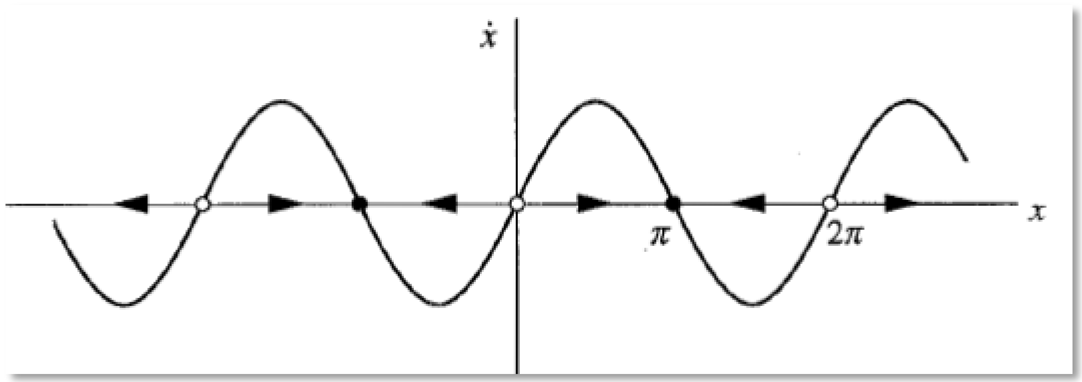
- We look at the "phase portrait": geometrically, picture of all possible trajectories (without solving the ODE analytically).

- Imagine: x is the position of an imaginary particle restricted to move in the line, and dx/dt is its velocity.

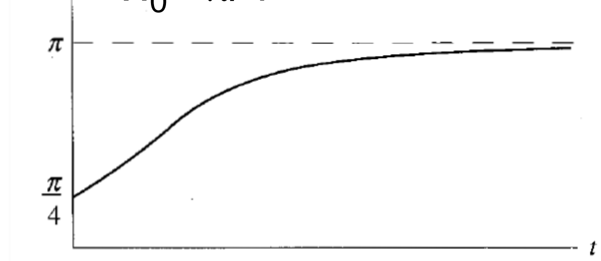
Imaginary particle moving in the horizontal axis

$\dot{x} = \sin x$

Flow to the right when $\dot{x} > 0$
Flow to the left when $\dot{x} < 0$

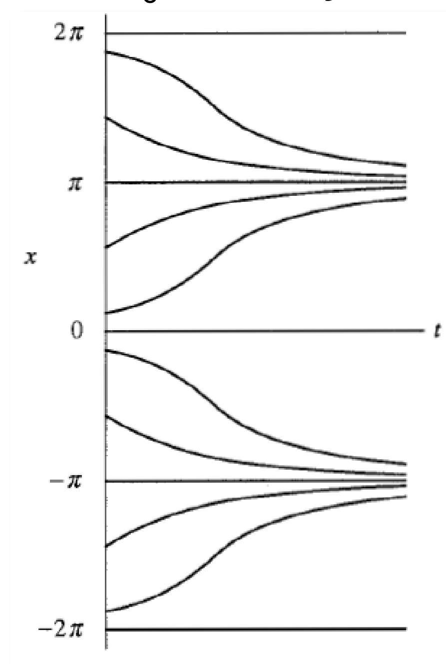


$x_0 = \pi/4$

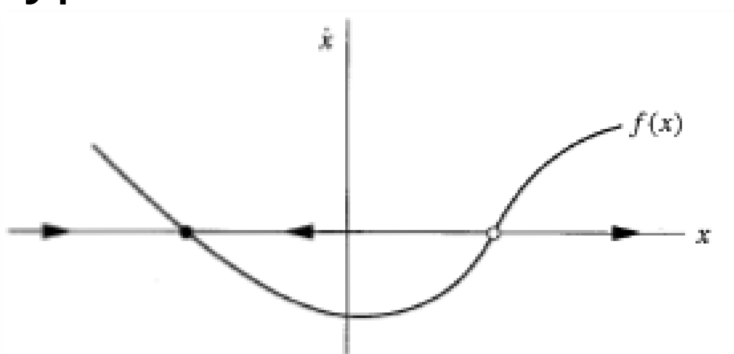


$\dot{x} = 0$ "Fixed points"

x_0 arbitrary



Two types of FPs: stable & unstable



Fixed points

$$\dot{x} = f(x) \qquad f(x^*) = 0$$

$x = x^*$ initially, then $x(t) = x^*$ for all time

$$\dot{x} = \sin x$$

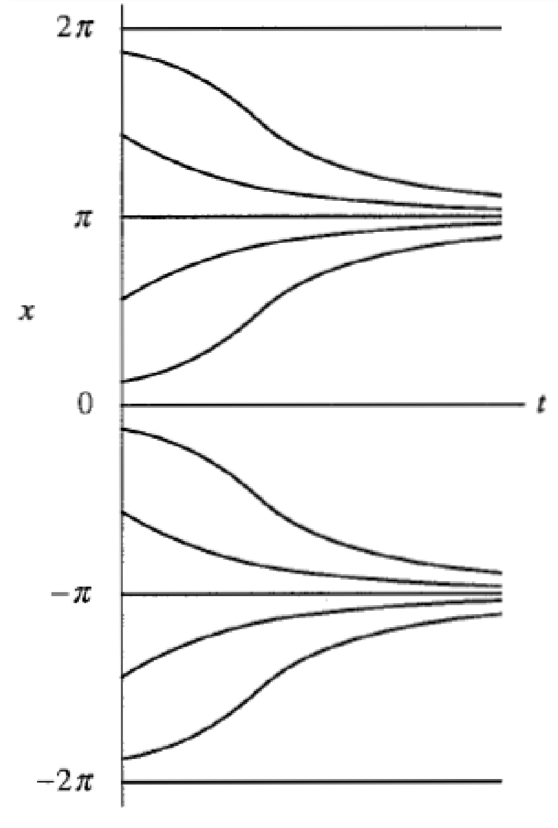
Fixed points = equilibrium solutions

- Stable (attractor or sink): nearby trajectories are attracted

  π and $-\pi$

- Unstable: nearby trajectories are repelled

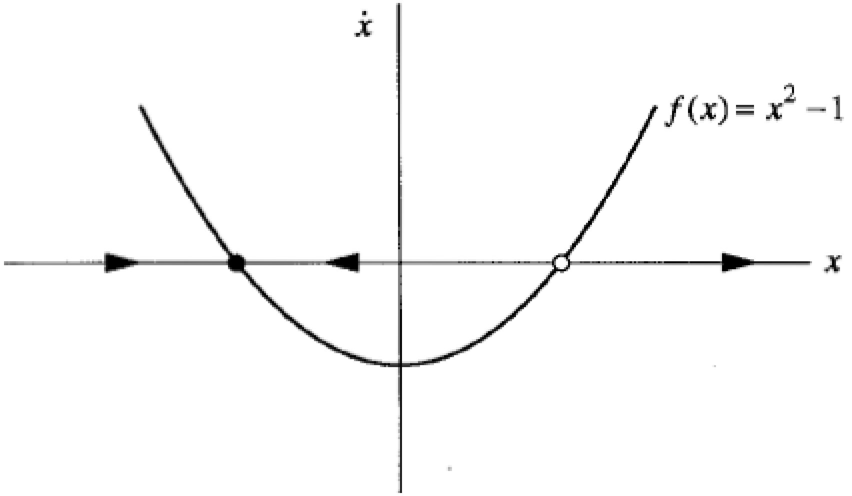
  0 and $\pm 2\pi$



Example 1

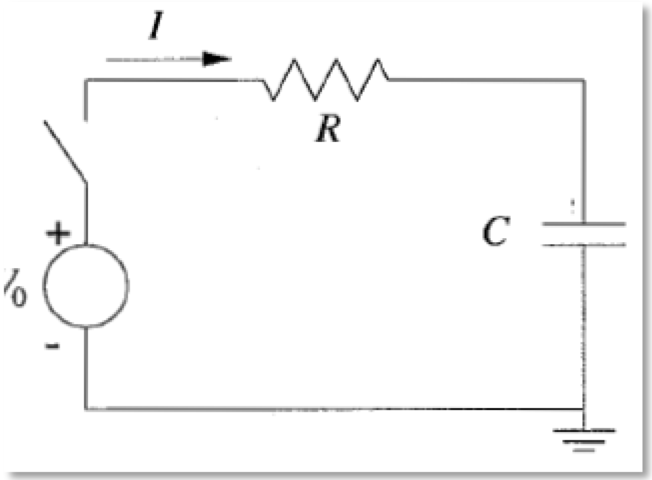
$$\dot{x} = x^2 - 1$$

- Find the fixed points and classify their stability



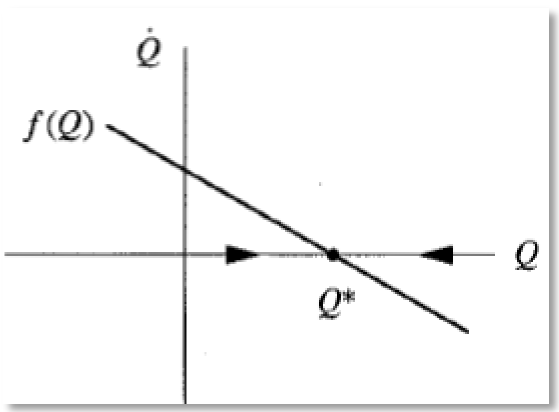
$x^* = -1$ is stable, and $x^* = 1$ is unstable

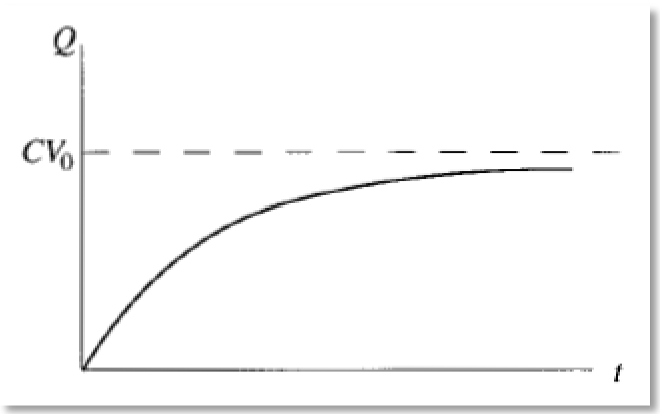
Example 2



$$-V_0 + R\dot{Q} + Q/C = 0$$

$$\dot{Q} = f(Q) = \frac{V_0}{R} - \frac{Q}{RC}$$





Example 3: population model for single species (e.g., bacteria)

- N(t): size of the population of the species at time t

$$\frac{dN}{dt} = \text{births} - \text{deaths} + \text{migration}$$

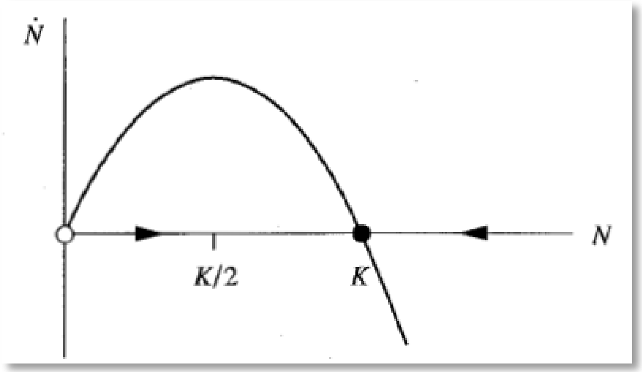
- Simplest model (Thomas Malthus 1798): no migration, births and deaths are proportional to the size of the population

$$\frac{dN}{dt} = bN - dN \quad \Rightarrow \quad N(t) = N_0 e^{(b-d)t}$$

Exponential grow!

More realistic model: logistic equation

- To account for limited food (Verhulst 1838): $\dot{N} = rN\left(1 - \frac{N}{K}\right)$

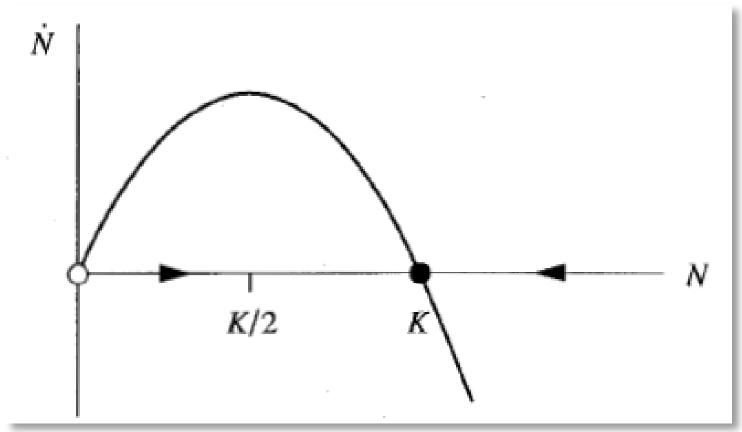


- If $N>K$ the population decreases
- If $N<K$ the population increases

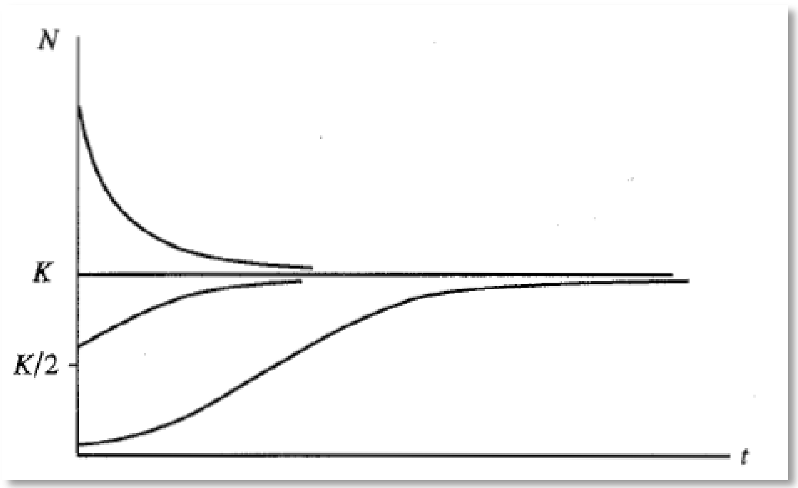
- K = "carrying capacity"

- The carrying capacity of a biological species in an environment is the maximum population size of the species that the environment can sustain indefinitely, given the food, habitat, water, etc.

How does a population approach the carrying capacity?



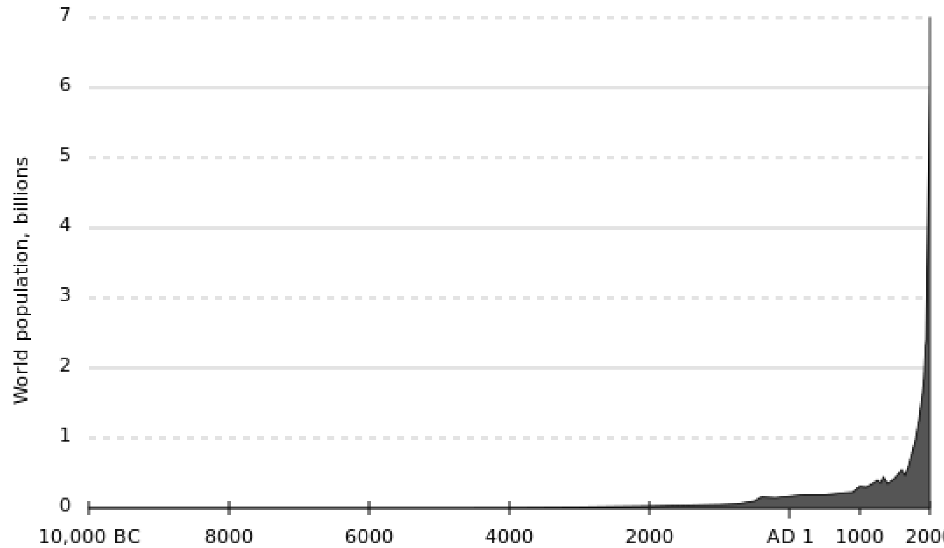
$$\dot{N} = rN\left(1 - \frac{N}{K}\right)$$

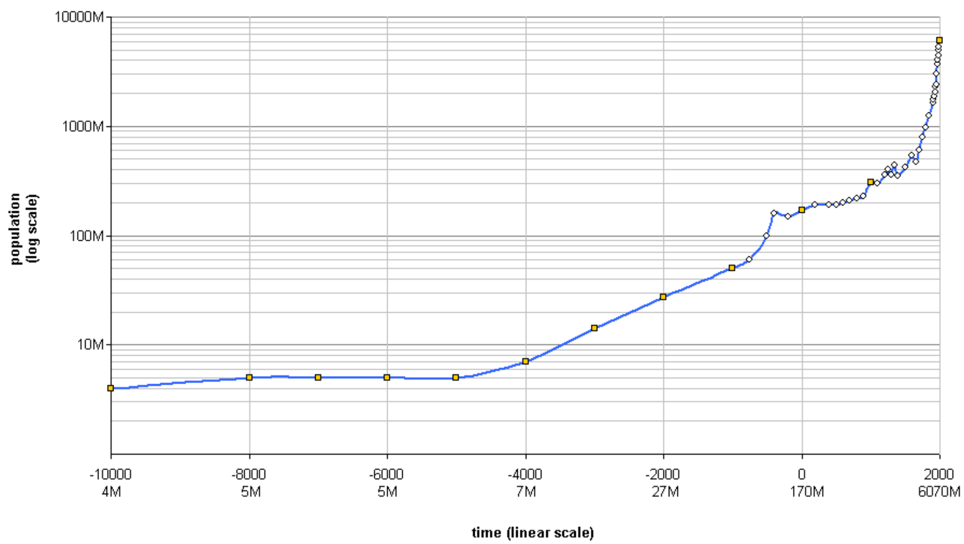


- Exponential or sigmoid approach.
- Good model only for simple organisms that live in constant environments.

And the human population?

Hyperbolic grow !
Technological advance
→ increase in the carrying capacity of land for people
→ demographic growth
→ more people
→ more potential inventors
→ acceleration of technological advance
→ accelerating growth of the carrying capacity…





Source: wikipedia

Linearization close to a fixed point

$\dot{x} = f(x)$ $\quad f(x^*) = 0$ $\quad \eta(t) = x(t) - x^*$ $\quad \eta$ = tiny perturbation

$$\dot{\eta} = \tfrac{d}{dt}(x - x^*) = \dot{x}$$

$$\dot{\eta} = \dot{x} = f(x) = f(x^* + \eta)$$

Taylor expansion $\quad f(x^* + \eta) = f(x^*) + \eta f'(x^*) + O(\eta^2)$

$f(x^*) = 0 \quad \boxed{\dot{\eta} = \eta f'(x^*) + O(\eta^2)}$

The slope *f'(x*)* at the fixed point determines the stability

$f'(x^*) > 0$ the perturbation η grows exponentially

$f'(x^*) < 0$ the perturbation η decays exponentially

$f'(x^*) = 0$ Second-order terms can not be neglected and a nonlinear stability analysis is needed.

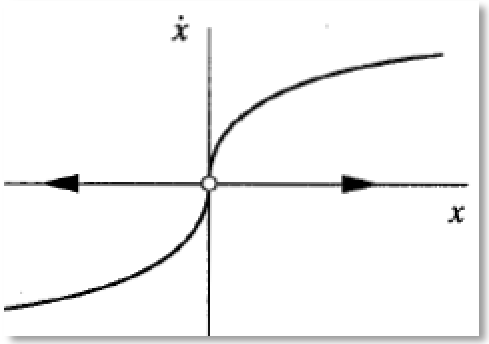
Bifurcation (more latter)

$1/|f'(x^*)|$ Characteristic time-scale

Existence and uniqueness

the solution to $\dot{x} = x^{1/3}$ starting from $x_0 = 0$ is *not* unique.

$$x(t) = 0 \qquad\qquad x(t) = \left(\tfrac{2}{3}t\right)^{3/2}$$



- Problem: *f '(0)* infinite

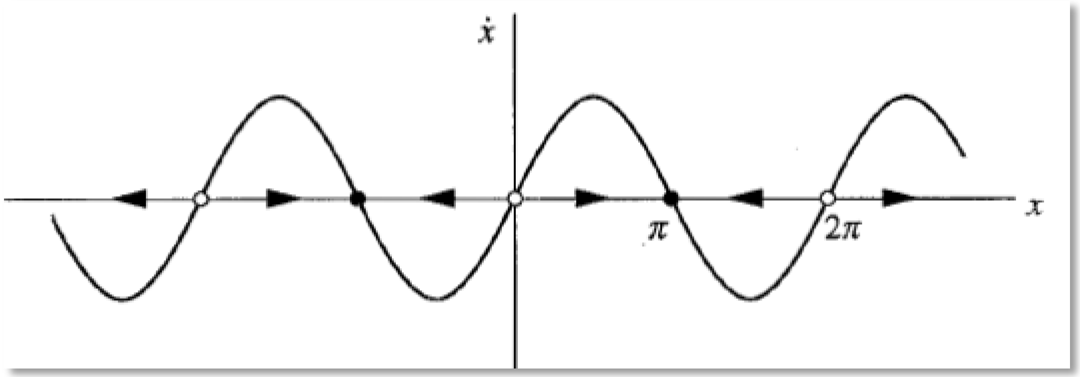
- When the solution of *dx/dt = f(x)* with *x(0)* = x_0 exists and is unique?
- Short answer: if *f(x)* is "well behaved", then a solution exists and is unique.
- "well behaved"?
- *f(x)* and *f '(x)* are both continuous on an interval of x-values and that x_0 is a point in the interval.
- Details: see Strogartz section 2.3.

Example 1

- Linear stability of the fixed points of $\dot{x} = \sin x$

$$x^* = k\pi$$

$$f'(x^*) = \cos k\pi = \begin{cases} 1, & k \text{ even} \\ -1, & k \text{ odd}. \end{cases}$$



- Stable: π and $-\pi$
- Unstable: 0, $\pm\, 2\pi$

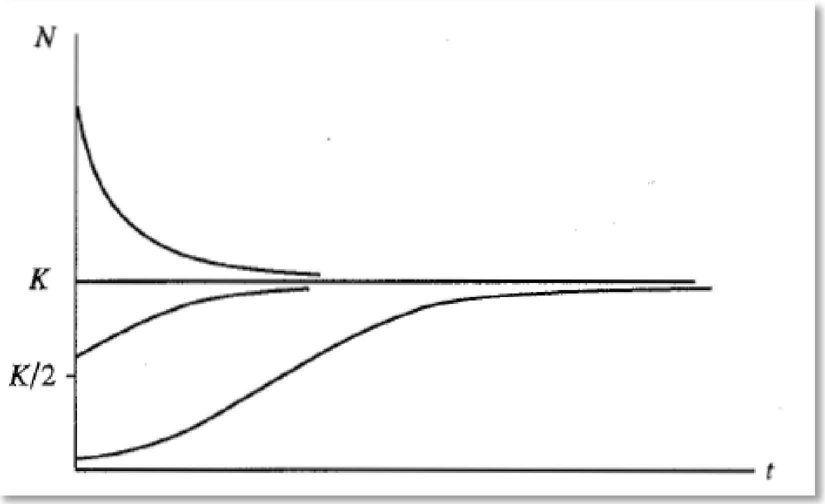
Example 2

- Logistic equation

$$\dot{N} = rN\left(1 - \frac{N}{K}\right)$$

$$N^* = 0 \text{ and } N^* = K$$

$$f'(0) = r \text{ and } f'(K) = -r \quad \Rightarrow \quad \begin{array}{l} N^* = 0 \text{ is unstable} \\ N^* = K \text{ is stable} \end{array}$$

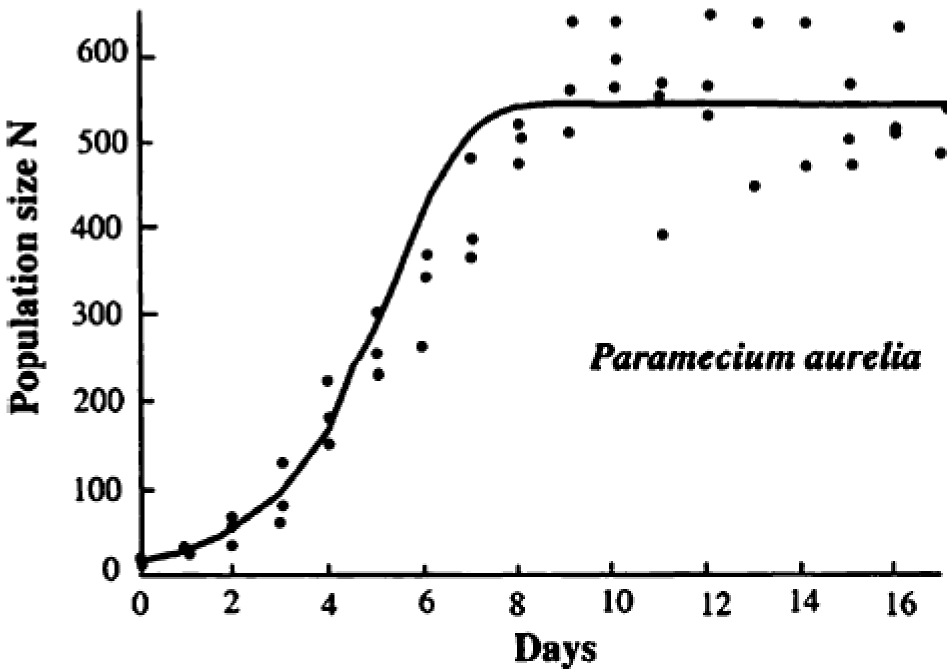


The two fixed points have the same characteristic time-scale:

$$1/|f'(N^*)| = 1/r$$

Good agreement with controlled population experiments

$$\dot{N} = rN\left(1 - \frac{N}{K}\right)$$

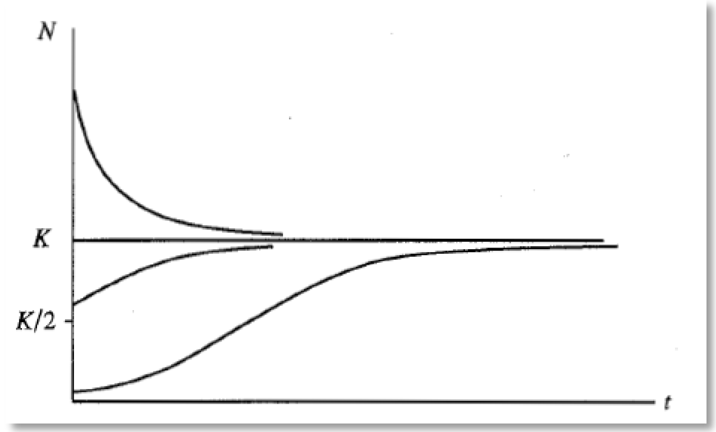


The population growth of the protozoan *Paramecium* in test tubes is a typical example (Figure 1.5). Under the conditions of the experiment, the population stopped growing when there were about 552 individuals per 0.5 ml. The time points show some scatter, which is caused both by the difficulty in accurately measuring population size (only a subsample of the population is counted) and by environmental variations over time and between replicate test tubes. A linear regression of the data N'/N versus N gives $r = 0.99$ and $K = 552$.

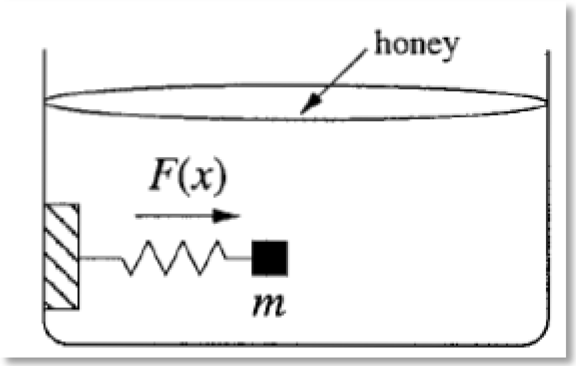
Lack of oscillations

$$\dot{x} = f(x)$$

General observation: only sigmoidal or exponential behavior, the approach is monotonic, **no oscillations**



Analogy:



$$m\ddot{x} + b\dot{x} = F(x)$$

Strong damping $\quad b\dot{x} >> m\ddot{x}$

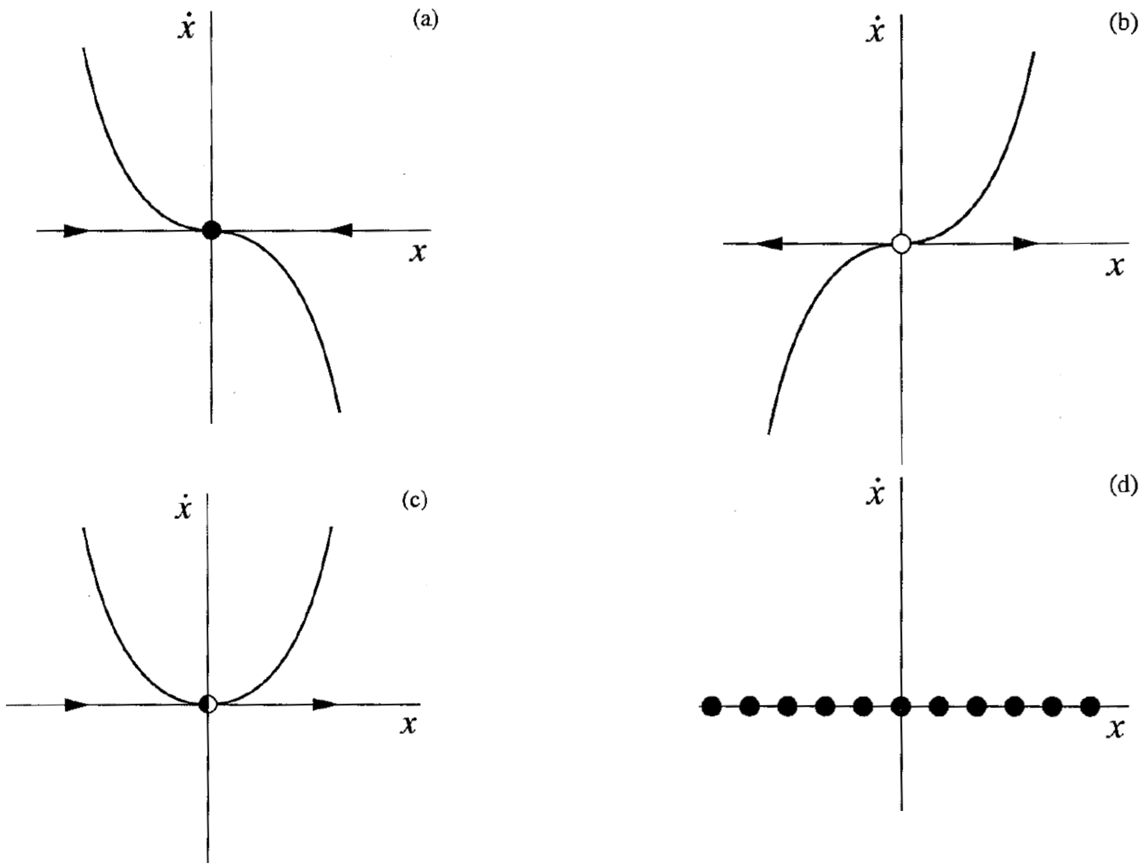
(over damped limit) $\quad b\dot{x} = F(x)$

To observe oscillations we need to keep the second derivative (weak damping).

Stability of the fixed point x* when f '(x*)=0?

(a) $\dot{x} = -x^3$ (b) $\dot{x} = x^3$ (c) $\dot{x} = x^2$ (d) $\dot{x} = 0$

In all these systems: $x^* = 0$ with $f'(x^*) = 0$

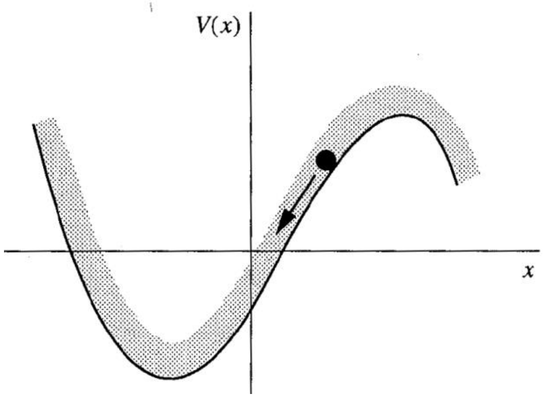


When f'(x*) = 0 nothing can be concluded from the linearization but these plots allow to see what goes on.

Potentials

$$\dot{x} = f(x) \qquad f(x) = -\frac{dV}{dx} \qquad \frac{dx}{dt} = -\frac{dV}{dx}$$

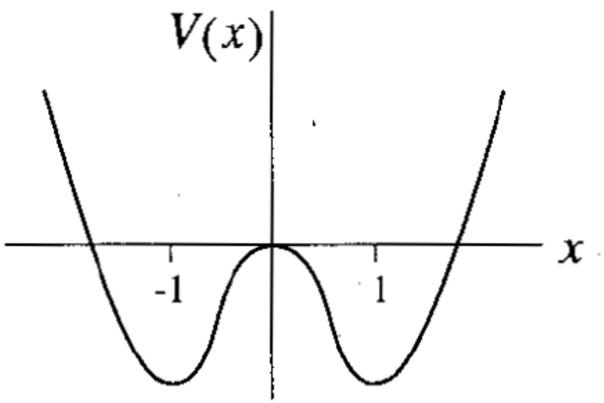
$$\frac{dV}{dt} = \frac{dV}{dx}\frac{dx}{dt} \qquad \frac{dV}{dt} = -\left(\frac{dV}{dx}\right)^2 \leq 0$$



V(t) decreases along the trajectory.

- Example: $\dot{x} = x - x^3$

$$V = -\tfrac{1}{2}x^2 + \tfrac{1}{4}x^4 + C$$



Two fixed points: x=1 and x=-1 *(Bistability)*.

Summary

- Flows on the line = first-order ODE: $dx/dt = f(x)$

- Fixed point solutions: $f(x^*) = 0$
  - stable if $f'(x^*) < 0$
  - unstable if $f'(x^*) > 0$
  - neutral (bifurcation point) if $f'(x^*) = 0$

- There are no periodic solutions; the approach to a fixed point is monotonic (sigmoidal or exponential).

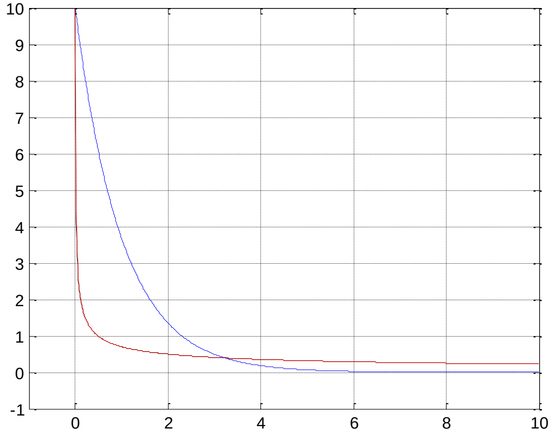
Class and homework

2.4.9 (Critical slowing down) In statistical mechanics, the phenomenon of "critical slowing down" is a signature of a second-order phase transition. At the transition, the system relaxes to equilibrium much more slowly than usual. Here's a mathematical version of the effect:

a) Obtain the analytical solution to $\dot{x} = -x^3$ for an arbitrary initial condition. Show that $x(t) \to 0$ as $t \to \infty$, but that the decay is not exponential. (You should find that the decay is a much slower algebraic function of t.)

b) To get some intuition about the slowness of the decay, make a numerically accurate plot of the solution for the initial condition $x_0 = 10$, for $0 \le t \le 10$. Then, on the same graph, plot the solution to $\dot{x} = -x$ for the same initial condition.

$$x(t) = \frac{x(0)}{\sqrt{1 + 2tx^2(0)}}$$



Outline

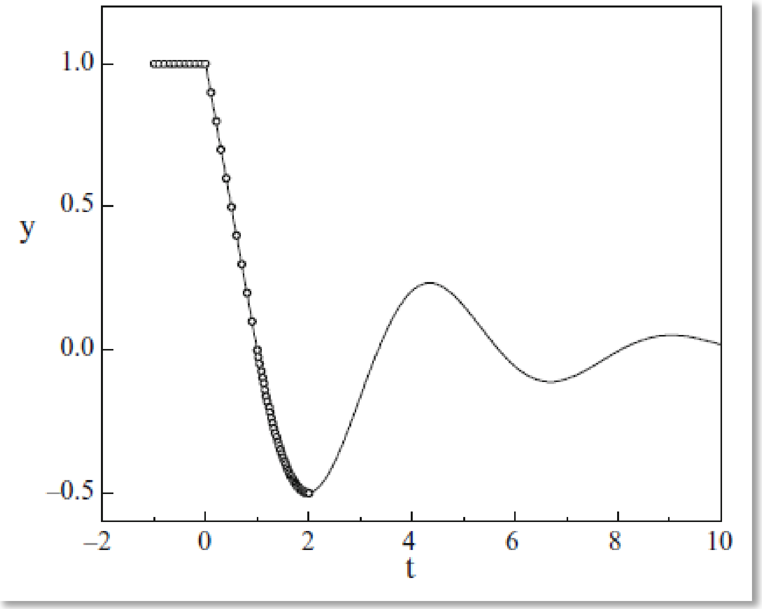
- Introduction to dynamical systems
- Introduction to flows on the line
- Solving equations with computer
- Fixed points and linear stability
- **Feedback control: delay differential equations**

Exception to no oscillations: delay differential equations (DDEs)

- Any system involving a **feedback control** will almost certainly involve time delays.
- In a 2D system delayed feedback can reduce oscillations, but in a 1D system it can induce oscillations.
- Example:

$$\frac{dy}{dt} = ky(t-\tau), \quad y(t) = 1 \quad \text{when} \quad -\tau \leq t < 0$$

  - Linear system
  - Infinite-dimensional system
  - Delay-induced oscillations.



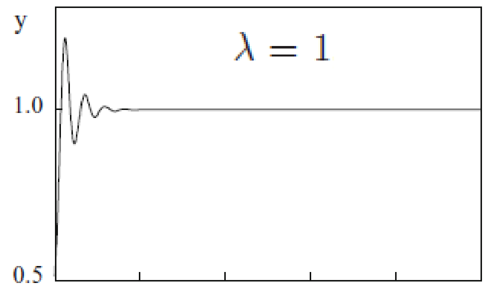
Example: population dynamics
Delayed logistic equation

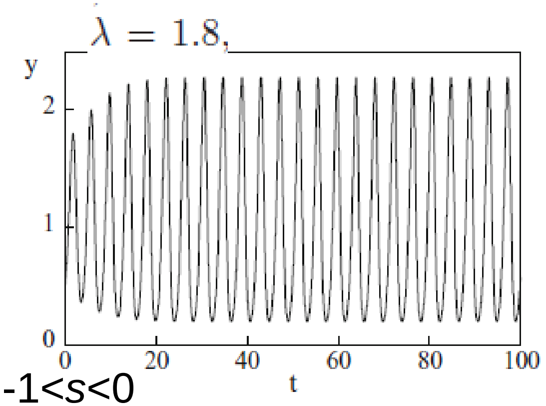
$$\frac{dN}{dt'} = rN\left(1 - \frac{N(t'-\tau)}{K}\right)$$

$$y \equiv N/K \quad \text{and} \quad t \equiv t'/\tau$$

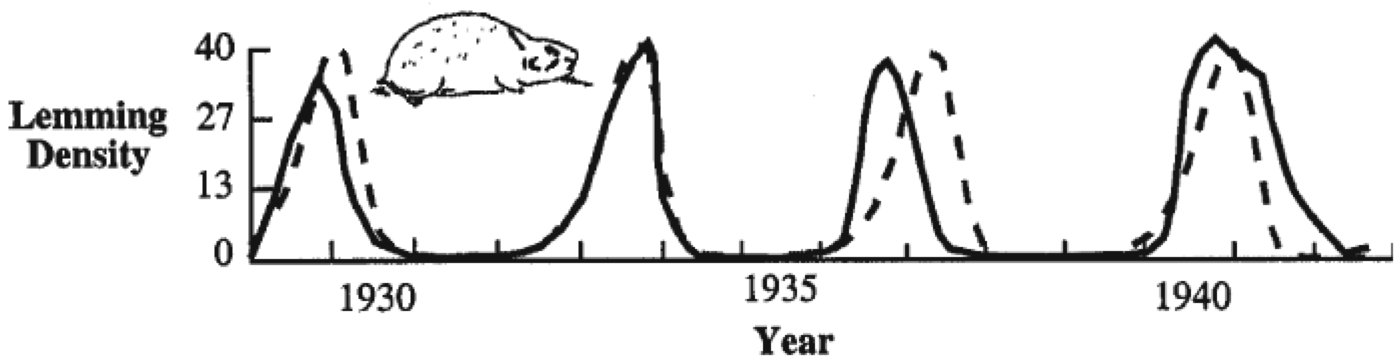
$$\frac{dy}{dt} = \lambda y(1 - y(t-1))$$

$$\lambda \equiv r\tau$$





The initial function is y=0.5 in -1<s<0



Lemming population cycles in the arctic north are nicely described by the logistic DDE with $r = 3.333/\text{yr}$ and $\tau = 9$ months ($\lambda = 3.333 \times 9/12 = 2.5$

- In a single-species population, the incorporation of a delay allows to explain the oscillations, without the predatory interaction of other species.

Example: Container crane
Delayed feedback control

It is important for the crane to move payloads rapidly and smoothly. If the gantry moves too fast the payload may start to sway, and it is possible for the crane operator to lose control of the payload.

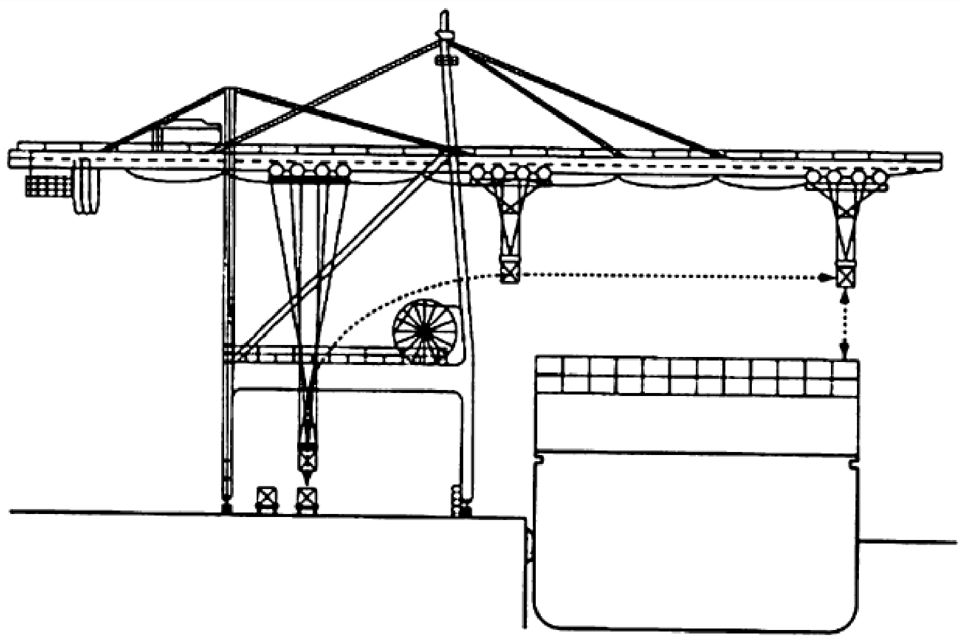


Figure 1.15: Container crane and ship (from H. Park and K.-S. Hong [181]).

Pendulum model for the crane, *y* represents the angle

$$y'' + \varepsilon y' + \sin(y) = -k\cos(y)(y(t-\tau) - y)$$

weakly damped oscillator
(not first-order equation,
without control payload
oscillations are possible)

Feedback control

Reduction of payload oscillations: why delayed feedback works?

Near the equilibrium solution *y*=0: $y'' + \varepsilon y' + y = -k(y(t-\tau) - y)$

Small delay: $y(t-\tau) \sim y - \tau y'$

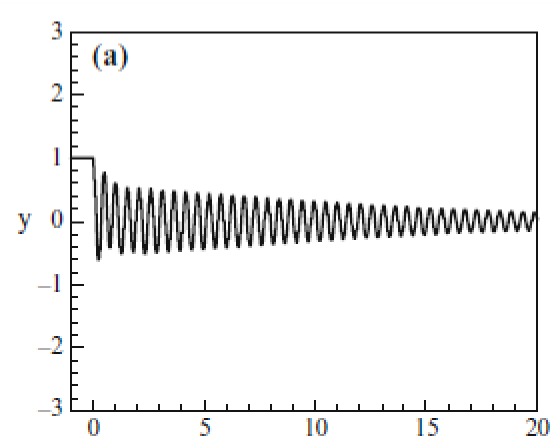
$$y'' + (\varepsilon - k\tau)y' + y = 0$$

The delay increases the damping.

Therefore: the oscillations decay faster.

but

- Small perturbation



- Large perturbation

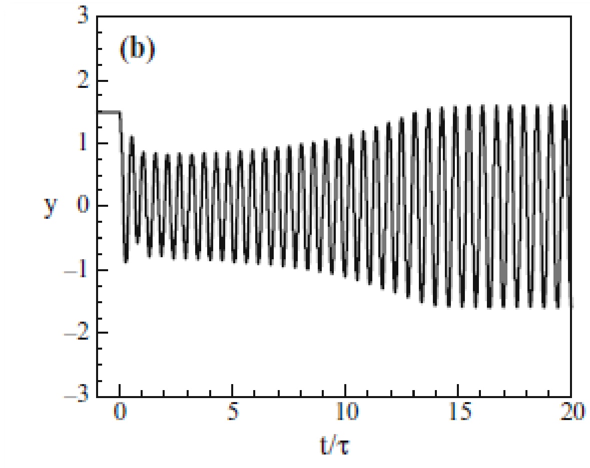


Figure 1.16: The values of the fixed parameters are $\tau = 12$, $\varepsilon = 0.1$, and $k = -0.15$. (a): $y'(0) = 0$ and $y = 1$ $(-\tau < t < 0)$; (b): $y'(0) = 0$ and $y = 1.5$ $(-\tau < t < 0)$.

Example: Car following model

$$x''_{n+1}(t+\tau) = \alpha(x'_n - x'_{n+1})$$

can be used for determining the location and speed of the following car (at $x = x_{n+1}$) given the speed pattern of the leading vehicle (at $x = x_n$). If a driver reacts too strongly (large value of α representing excessive braking) or too late (long reaction time τ), the spacing between vehicles may become unstable (i.e., we note damped oscillations in the spacing between vehicles).

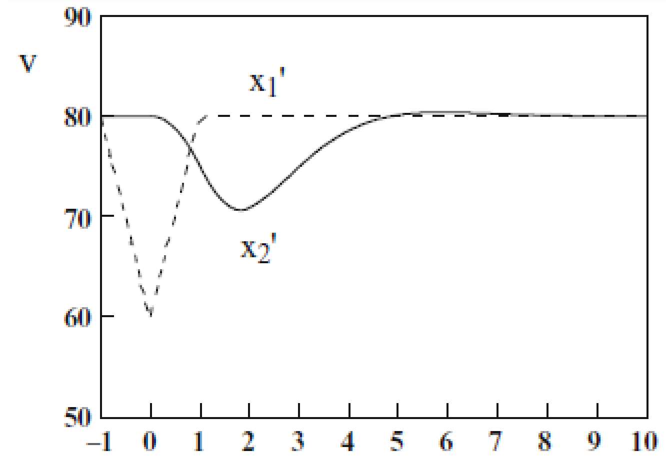
Typical solution for two cars

$$x''_{n+1}(t+\tau) = \alpha(x'_n - x'_{n+1})$$

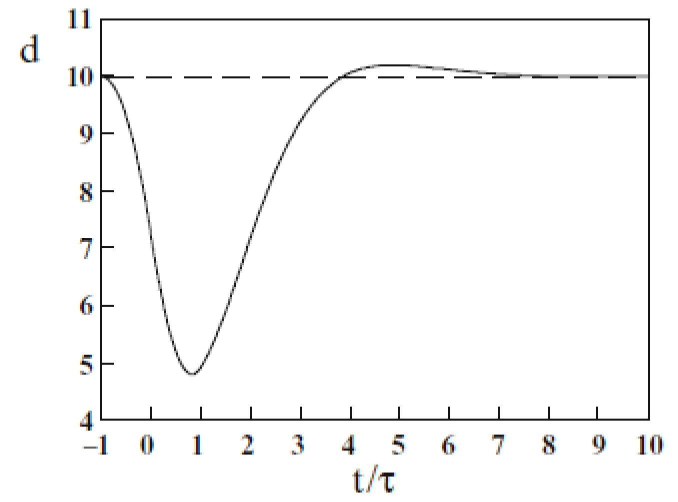
The lead vehicle reduces its speed of 80 km/h to 60 km/h and then accelerates back to its original speed. The initial spacing between vehicles is 10 m.

$\alpha = 0.5\ \text{s}^{-1}$ and $\tau = 1$ s

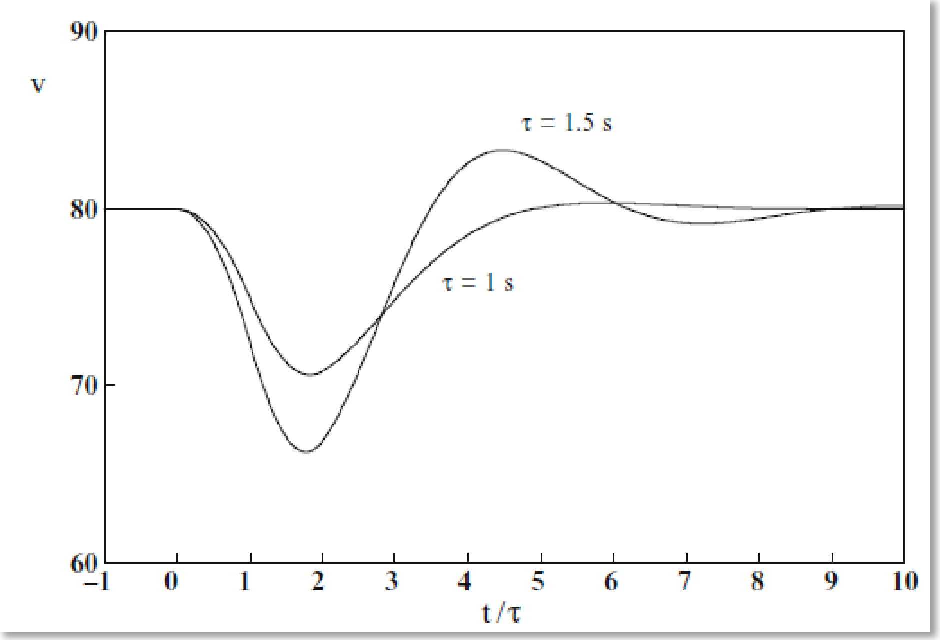
- Speed of the two cars



- Distance between the two cars



Alcohol effect



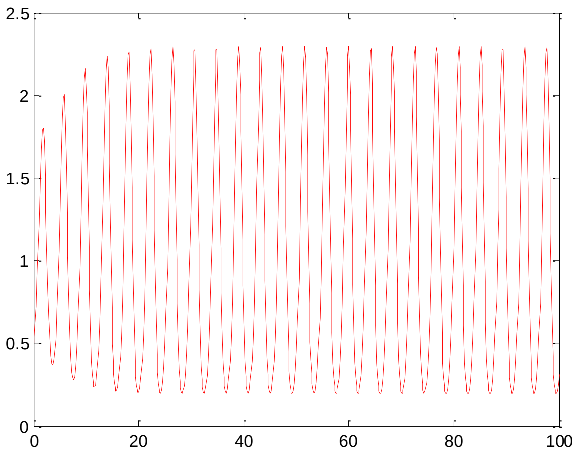
A sober driver needs about 1 s in order to start breaking in view of an obstacle.
With 0.5 g/l alcohol in blood (2 glasses of wine), this reaction time is estimated to be about 1.5 s.
$\Rightarrow$ oscillations near the stable equilibrium increase.

Solving DDEs

Example 1: Delayed logistic equation $\frac{dy}{dt} = \lambda y(1 - y(t-1))$

ic =constant initial function

```
function solve_delay1
tau = 1;
ic = [0.5];
tspan = [0 100];
h = 1.8;
sol = dde23(@f,tau,ic,tspan);
plot(sol.x,sol.y(1,:),'r-')
function v=f(t,y,Z)
v = [h*y(1).*(1-Z(1))];
end
end
```

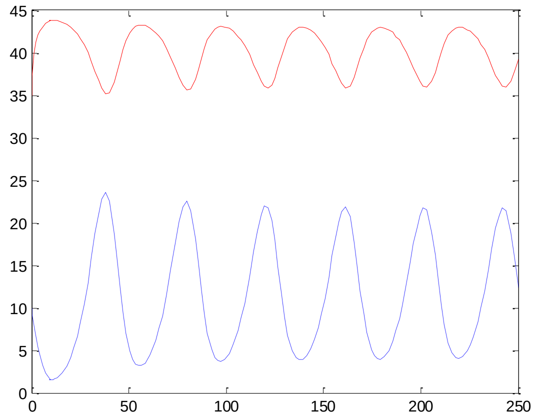


Solving DDEs

Example 2: Prey (x) and predator (y) model

$$\frac{dx}{dt} = x(t)\left\{2\left[1 - \frac{x(t)}{50}\right] - \frac{y(t)}{x(t)+40}\right\} - 10,$$
$$\frac{dy}{dt} = y(t)\left[-3 + \frac{6x(t-\tau)}{x(t-\tau)+40}\right].$$

```
function solve_delay2
tau=9;
ic = [35;10];
tspan = [0 250];
h = 10;
sol = dde23(@f,tau,ic,tspan);
plot(sol.x,sol.y(1,:),'r-',sol.x,sol.y(2,:),'b--')
function v=f(t,y,Z)
v = [y(1)*(2*(1-y(1)/50)-y(2)/(y(1)+40))-h
    y(2)*(-3+6*Z(1)/(Z(1)+40))];
end
end
```



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