Nonlinear systems, chaos and control in Engineering

Module 1
One-dimensional systems

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Schedule

Flows on the line
(Strogatz ch.1 & 2)
18/11 (3 hs)
- Introduction
- Solving equations with computer
- Fixed points and linear stability
- Feedback control and delays

Bifurcations
(Strogatz ch. 3)
18/11 (2 hs) & 24/11 (3 hs)
- Introduction
- Saddle-node
- Transcritical
- Pitchfork
- Examples

Flows on the circle
(Strogatz ch. 4)
25/11 (2 hs)
- Introduction to phase oscillators
- Nonlinear oscillator
- Fireflies and entrainment
Flows on the line: outline

- Introduction to dynamical systems
- Introduction to flows on the line
- Solving equations with computer
- Fixed points and linear stability
- Feedback control: delay differential equations
- Systems that evolve in time.
- Examples:
  - Pendulum clock
  - Neuron
- Dynamical systems can be:
  - linear or nonlinear (harmonic oscillator – pendulum);
  - deterministic or stochastic;
  - low or high dimensional;
  - continuous time or discrete time.

In this course: nonlinear systems (Nonlinear Dynamics)
Figure 2 a, The membrane potential of a single neuron in the cat visual cortex. A visual stimulus causes a sustained barrage of synaptic input, which triggers three cycles of depolarization. The first cycle does not reach the threshold for generating a spike, but the second and third do. (Data provided by B. Ahmed and K. Martin.) b, Variability of the firing response of a single neuron in monkey cortex (each line in the trace corresponds to a spike of the type shown in a). The same stimulus is presented five times and triggers about 40 spikes each time; yet the exact timing of individual spikes shows random variation. (Data provided by W. Newsome and K. Britten.)
Given the initial condition: possible evolution

• The neuron settles down to equilibrium (rest state or “fixed point”). – module 1 in this course

• Keeps spiking in cycles (“limit cycle”). – module 2

• More complicated: **chaotic** or **complex** evolution (“chaotic attractor”). – module 3
Historical development of the *Theory of Dynamical Systems*
Isaac Newton: studied planetary orbits and solved analytically the “two-body” problem (earth around the sun).

Since then: a lot of effort for solving the “three-body” problem (earth-sun-moon) – Impossible.
Christiaan Huygens (mid-1600s, Dutch mathematician)

- Patented the first pendulum clock.
- Observed the synchronization of two clocks.

http://www.youtube.com/watch?v=izy4a5erom8
Henri Poincare (French mathematician). Instead of asking “which are the exact positions of planets (trajectories)?” he asked: “is the solar system stable for ever, or will planets eventually run away?”

He developed a geometrical approach to solve the problem.

Introduced the concept of “phase space”.

He also had an intuition of the possibility of chaos:
Deterministic system: the present state (initial condition) fully determines the future state. There is no randomness but the system can be unpredictable.

Poincare: “The evolution of a **deterministic** system can be aperiodic, unpredictable, and strongly depends on the initial conditions”
Computers drive economic growth and transform how we live and work.

Computes allowed to experiment with equations.

Powerful tool to advance the “Theory of Dynamical Systems”.


Intuition of chaotic motion on a strange attractor.

He also showed that there is structure and order in chaotic motion.
The Lorentz system

Lorentz studied meteorological prediction using Navier-Stokes simplified equations:

\[
\begin{align*}
\dot{x} &= \delta (x - y) \\
\dot{y} &= x(r - z) - y \\
\dot{z} &= xy - bz
\end{align*}
\]

- **3 variables:**
  - \( x \): rotation rate of a cylindrical mass of gas,
  - \( y \): thermal gradient,
  - \( z \): temperature variation.

- **3 Parameters:**
  - \( \delta \): ratio between viscosity and thermal conductivity (Prandtl number),
  - \( R \): temperature difference between top and bottom of cylinder (Rayleigh number),
  - \( b \): ratio between width and height of the cylinder.
Starting from an initial condition \((x_0, y_0, z_0)\) by numerically integrating the equations we can plot the trajectory in the *phase space* (Lorentz’s Attractor).

Lorentz found extreme sensitivity to initial conditions \(\Rightarrow\) impossibility of long-term meteorological predictions.
Order within chaos and self-organization

- **Ilya Prigogine** (Belgium, born in Moscow, Nobel Prize in Chemistry 1977)
- Thermodynamic systems far from equilibrium.
- Discovered that, in chemical systems, the interplay of (external) **input of energy** and **dissipation** can lead to “self-organised” patterns.
- Reverse the rule of maximization of entropy (second law of thermodynamics).
- Wide implications to biological systems and the evolution of life.
One-dimensional spatio-temporal patterns

Patterns in nature
(source: wikipedia)
## Entropy (H) and Complexity (C)

<table>
<thead>
<tr>
<th>Complete order</th>
<th>Chaos</th>
<th>Complete disorder</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Complete order image" /></td>
<td><img src="image2.png" alt="Chaos image" /></td>
<td><img src="image3.png" alt="Complete disorder image" /></td>
</tr>
</tbody>
</table>

- **Complete order**
  - $H = 0$
  - $C = 0$

- **Chaos**
  - $H \neq 0$
  - $C \neq 0$

- **Complete disorder**
  - $H = \text{Max}$
  - $C = 0$

O. A. Rosso (2009)
Newtonian physics has been extended three times:

- First, with the use of the wave function in quantum mechanics.
- Then, with the introduction of space-time in relativity.
- And finally, with the recognition of indeterminism in nonlinear systems.

*Chaos is the third great revolution of 20th-century physics, after relativity and quantum theory.*

In the 1960’s he did experiments trying to understand the effects of perturbations in biological clocks (circadian rhythms).

What is the effect of an external perturbation on subsequent oscillations?
He studied the periodic emergence of a fruit fly that as a $\approx 24$-hour rhythmic emergency.

Using brief pulses of light, found that the periodic emergence of the flies was shifted, and the shift depended on the timing and the duration of the light pulse.

Also found that there is a critical timing and duration that results in no further periodic emergency (destroys the biological clock).

The work has wide implications, for example, for cardiac tissue: some cardiac failures are related to perturbed oscillations.
Robert May (Australian, 1936): population biology


\[ x_{t+1} = f(x_t) \]

Example: \( f(x) = r x(1 - x) \)

Difference equations ("iterated maps"), even though simple and deterministic, can exhibit different types of dynamical behaviors, from stable points, to a bifurcating hierarchy of stable cycles, to apparently random fluctuations.
Models of dynamical systems

Continuous-time ordinary differential equations (ODEs)

\[ \frac{dx}{dt} = \dot{x} = f(x, r) \]

Discrete-time equations (iterated maps)

\[ x_{t+1} = f(x_t, r) \]

\( r = \text{control parameter(s)} \)

Example: Lorentz model

\[
\begin{align*}
\dot{x} &= \delta(x - y) \\
\dot{y} &= x(r - z) - y \\
\dot{z} &= xy - bz \\
r &= (\delta, r, b)
\end{align*}
\]

Example: Logistic map

\[ x_{t+1} = r \, x_t (1 - x_t) \]
The logistic map

\[ x(i + 1) = r \cdot x(i) [1 - x(i)] \]

- **r = 2.8**, Initial condition: \( x(1) = 0.2 \)
  - Transient relaxation → long-term stability

- **r = 3.3**, **r = 3.5**, **r = 3.9**
  - Transient dynamics → stationary oscillations (regular or irregular)
  - “period-doubling” bifurcations to chaos

Parameter \( r \)
In 1975, Mitchell Feigenbaum (American mathematical physicist), using a small HP-65 calculator, discovered the scaling law of the bifurcation points

\[
\lim_{n \to \infty} \frac{r_{n-1} - r_{n-2}}{r_n - r_{n-1}} = 4.6692...
\]

Then, he provided a mathematical proof (by using the “renormalization concept” – connecting to phase transitions in statistical physics).

Then, he showed that the same behavior, with the same mathematical constant, occurs within a wide class of functions, prior to the onset of chaos (universality).

Very different systems (in chemistry, biology, physics, etc.) go to chaos in the same way, quantitatively.
The first magnetic card-programmable handheld calculator

HP-65 in original hard case with manuals, software "Standard Pac" of magnetic cards, soft leather case, and charger
Benoit Mandelbrot (Polish-born, French and American mathematician 1924-2010): “self-similarity” and fractal objects: each part of the object is like the whole object but smaller.

Because of his access to IBM's computers, Mandelbrot was one of the first to use computer graphics to create and display fractal geometric images.
Are characterized by a “fractal” dimension that measures roughness (more in Module 3)

- Romanesco broccoli $D=2.66$
- Human lung $D=2.97$
- Coastline of Ireland $D=1.22$

Video: http://www.ted.com/talks/benoit_mandelbrot_fractals_the_art_of_roughness#t-149180
In the 80’s: can we observe chaos experimentally?

- **Optical chaos**: first observed in laser systems.

... the optically pumped FIR laser system. Key work has been contributed by Vilaseca, Corbalan, and coworkers,\(^4\) Khanin and colleagues,\(^5\) and by Harrison, Moloney, and coworkers.\(^6\) However, the FIR laser experiments are even ...

N. B. Abraham, OPN 1989

More latter about experiments with optical chaos in our lab.
In the 90’s: can we control chaotic dynamics?

- **Ott, Grebogi and Yorke** (1990)
  Unstable periodic orbits can be used for control: wisely chosen periodic kicks can maintain the system near the desired orbit.

- **Pyragas** (1992)
  Control by using a continuous self-controlling feedback signal, whose intensity is practically zero when the system evolves close to the desired periodic orbit but increases when it drifts away.
Experimental demonstration of control of optical chaos

- Raj Roy and others (1994)
The 1990s: synchronization of two chaotic systems
Pecora and Carroll, PRL 1990

Unidirectionally coupled Lorenz systems: the ‘χ’ variable of the response system is replaced by the ‘χ’ variable of the drive system.

- **Drive system**
  \[
  \frac{dx_1}{dt} = -\sigma(y_1 - x_1) \\
  \frac{dy_1}{dt} = -x_1z_1 + rx_1 - y_1 \\
  \frac{dz_1}{dt} = x_1y_1 - bz_1
  \]

- **Response system**
  \[
  \frac{dy_2}{dt} = -x_1y_2 + rx_1 - y_2 \\
  \frac{dz_2}{dt} = x_1y_2 - bz_2
  \]

As time approaches infinity, \( |y_2 - y_1| \to 0 \) and \( |z_2 - z_1| \to 0 \).
In the case of low dimensional chaotic attractors, is possible to break the system and extract the message (G. Perez y H. Cerdeira, PRL 1995)

Transmission of secure information (Cuomo–Oppenheim, PRL 1993)

Is the system secure?

In the case of low dimensional chaotic attractors, is possible to break the system and extract the message (G. Perez y H. Cerdeira, PRL 1995)
Different type of coupling and different types of synchronization

\[ \frac{dx_1}{dt} = F(x_1) \]
\[ \frac{dx_2}{dt} = F(x_2) + \alpha E(x_1 - x_2) \]

• Complete (CS): \( x_1(t) = x_2(t) \) (identical systems)

• Phase (PS): the phases of the oscillations are synchronized, but the amplitudes are not.

• Lag (LS): \( x_1(t+t_0) = x_2(t) \)

• Generalized (GS): \( x_2(t) = f(x_1(t)) \) (f depends on the strength of the coupling)
Experimental observation with coupled lasers
Synchronization with time-delay

\[
\frac{dx}{dt} = f[x(t)]
\]

\[
\frac{dy}{dt} = f[y(t)] + k[x(t) - y(t - \tau)]
\]

- Solution: \( x(t) = y(t - \tau) \Rightarrow x(t + \tau) = y(t) \)

Is it possible to anticipate/predict the evolution of a chaotic system?

The solution is stable only for ‘small’ values of \( \tau \) (Voss, PRL 2001)
Experimental demonstration with electronic circuits

M. Cizak et al, PRL 2003
In the last decade: can we exploit / use chaotic dynamics?

An example from optical chaos

After processing the signal, **ultra-fast generation of arbitrarily long sequences of random bits.**

I. Kanter et al, Nature Photonics 2010
Dynamical systems in optics: Two examples from our lab

- **Laser spikes**: intensity vs. time
  
  ![Laser spikes graph]

  [https://youtu.be/nItBQG_IWQ](https://youtu.be/nItBQG_IWQ)

- **Neuronal spikes**

  ![Neuronal spikes graph]

- **Extreme pulses**

  ![Extreme pulses diagram]

  Interested? TFGs available, contact us.
From dynamical systems to complex systems
Interest moves from chaotic systems to complex systems (low vs. very large number of variables).

Networks (or graphs) of interconnected systems

**Complexity science**: dynamics of emergent properties
- Epidemics
- Rumor spreading
- Transport networks
- Financial crises
- Brain diseases
- Etc.
Synchronization of a large number of coupled oscillators

http://www.youtube.com/watch?v=DD7YDyF6dUk

Figure 1 | Fireflies, fireflies burning bright. In the forests of the night, certain species of firefly flash in perfect synchrony — here *Pteroptyx malaccae* in a mangrove apple tree in Malaysia. Kaka *et al.* and Mancoff *et al.* show that the same principle can be applied to oscillators at the nanoscale.
Kuramoto model
(Japanese physicist, 1975)

- Model of **all-to-all** coupled phase oscillators.

\[
\frac{d\theta_i}{dt} = \omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i) + \xi_i, \quad i = 1...N
\]

K = coupling strength, \( \xi_i \) = stochastic term (noise)

Describes the emergence of collective behavior (synchronization)

How to quantify synchronization?

With the order parameter:

\[
re^{i\psi} = \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j}
\]

\( r = 0 \) incoherent state (oscillators are scattered in the unit circle)
\( r = 1 \) all oscillators are in phase \((\theta_i = \theta_j \ \forall \ i,j)\)
Synchronization transition as the coupling strength increases

Strogatz
Nature 2001
The Kuramoto model allows for deriving exact results (Steven Strogatz and others, late 90’).

Video: https://www.ted.com/talks/steven_strogatz_on_sync
The challenge: to understand how the interplay of **structure and dynamics** (of individual units) results in emergent collective behavior.

Strogatz
Nature 2001,
Main feature of a network: the degree distribution

Liu et al, Nature 2011
Scientific cooperation: red FISES (A. Diaz-Guilera, UB)
Electric power grid (N. Rubido PhD Thesis 2014)
Generators and substations are shown as small blue bars. The lines connecting them are transmission lines and transformers. Line thickness and colour indicate the voltage level: red, 765 kV and 500 kV; brown, 345 kV; green, 230 kV; grey, 138 kV and below. Pink dashed lines are transformers.
Strogatz, Nature 2001
Graph consisting of the pages of a web site and their mutual hyperlinks, which are directed. Communities are indicated by the colors (Fortunato, 2010)
Nodes represent symbiotically connected species, such as plants and pollinators (Gao et al, Nature 2016)

Interacting networks (example: friendship relations in Facebook and twitter)
Transport networks

Fig. 2. Route map for Continental Airlines (courtesy of Continental Airlines).
The start of Network Theory: The Königsberg Bridge problem

The problem: to devise a walk through the city that would cross each bridge once and only once. The starting and ending points of the walk need not be the same.

By representing the network as a **Graph** (a set of “vertices” connected by a set of “edges”) Euler (1707-1783) proved that the problem has no solution.
Brain functional network

Eguiluz et al, PRL 2005
Chavez et al, PRE 2008
Nodes and links of the climate network
Time series of a climate variable (air temperature, wind, precipitation, etc.)

Similarity measure (correlation, mutual information, etc.)

Winds & ocean currents

Climate network

Threshold

Eguiluz et al, PRL 2005
Deza et al, Chaos 2013
Donges et al, Chaos 2015
The theory of dynamical systems allows to
- understand the dynamics of low-dimensional systems,
- to uncover “order within chaos”,
- universality features in the transition to chaos, and
- provides tools for controlling chaotic behavior.

Complexity science is aimed to understand the emerging phenomena in large sets of interacting systems.

Both, dynamical systems and complexity science have multiple applications, and involve the work of mathematicians, physicists, biologists, computer scientists, engineers, etc.
- Introduction to dynamical systems
- **Introduction to flows on the line**
- Solving equations with computer
- Fixed points and linear stability
- Feedback control: delay differential equations
Types of dynamical systems

- **Continuous time**: differential equations
  - Ordinary differential equations (ODEs). Example: damped oscillator
    \[
    m \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + kx = 0
    \]
  - Partial differential equations (PDEs). Example: heat equation
    \[
    \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}
    \]

- **Discrete time**: difference equations or “iterated maps”. Example: the logistic map
  \[
  x(i+1) = r \times x(i)[1-x(i)]
  \]
ODEs can be written as **first-order** differential equations

\[
m \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + kx = 0
\]

\[
\Rightarrow \quad \dot{x}_1 = f_1(x_1, \ldots, x_n)
\]

\[
\quad \vdots
\]

\[
\dot{x}_n = f_n(x_1, \ldots, x_n)
\]

\[
\dot{x} = f(x)
\]

- **First example: harmonic oscillator**

\[
x_1 = x \text{ and } x_2 = \dot{x}
\]

\[
\dot{x}_2 = \ddot{x} = -\frac{b}{m} \dot{x} - \frac{k}{m} x = -\frac{b}{m} x_2 - \frac{k}{m} x_1
\]

- **Second example: pendulum**

\[
\ddot{x} + \frac{g}{L} \sin x = 0
\]

\[
\Rightarrow \quad \dot{x}_1 = x_2
\]

\[
\dot{x}_2 = -\frac{g}{L} \sin x_1
\]
Trajectory in the phase space

Given the initial conditions, $x_1(0)$ and $x_2(0)$, we predict the evolution of the system by solving the equations: $x_1(t)$ and $x_2(t)$.

$x_1(t)$ and $x_2(t)$ are solutions of the equations.

The evolution of the system can be represented as a trajectory in the phase space.

$\Rightarrow$ two-dimensional (2D) dynamical system.

Key argument (Poincare): find out how the trajectories look like, without solving the equations explicitly.
Classification of dynamical systems described by ODEs (I/II)

\[ \dot{x} = f(x) + \xi(t) \]

- **f(x) linear**: in the function \( f \), \( x \) appears to first order only (no \( x^2 \), \( x_1x_2 \), \( \sin(x) \) etc.). Then, the behavior can be understood from the sum of its parts.
- **f(x) nonlinear**: superposition principle fails!
- Example of linear system: harmonic oscillator
  \[ m \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + kx = 0 \quad \Rightarrow \quad \begin{align*}
  \dot{x}_1 &= x_2 \\
  \dot{x}_2 &= -\frac{b}{m} x_2 - \frac{k}{m} x_1
  \end{align*} \]
  In the right-hand-side \( x_1 \) and \( x_2 \) appear to first power (no products etc.)
- Example of nonlinear system: pendulum
  \[ \dot{x} + \frac{g}{L} \sin x = 0 \quad \Rightarrow \quad \begin{align*}
  \dot{x}_1 &= x_2 \\
  \dot{x}_2 &= -\frac{g}{L} \sin x_1
  \end{align*} \]
Classification of dynamical systems described by ODEs (II/II)

\[ \dot{x} = f(x) + \xi(t) \]

- \( \xi = 0 \): deterministic.
- \( \xi \neq 0 \): stochastic (real life) – simplest case: additive noise.

- \( x \): vector with few variables (\( n < 4 \)): low dimensional.
- \( x \): vector with many variables: high dimensional.

- \( f \) does not depend on time: autonomous system.
- \( f \) depends on time: non-autonomous system.
Example of non-autonomous system: a forced oscillator

\[ m\ddot{x} + b\dot{x} + kx = F \cos t \]

- Can also be written as first-order ODE

\[
\begin{align*}
    x_1 &= x \text{ and } x_2 = \dot{x} \\
    x_3 &= t \quad \dot{x}_3 = 1
\end{align*}
\]

\[
\begin{align*}
    \dot{x}_1 &= x_2 \\
    \dot{x}_2 &= \frac{1}{m}(-kx_1 - bx_2 + F \cos x_3) \\
    \dot{x}_3 &= 1
\end{align*}
\]

- 3D system: to predict the future evolution we need to know the present state \((t, x, dx/dt)\).
So…what is a “flow on the line”?

- A one-dimensional autonomous dynamical system described by a first-order ordinary differential equation

\[ \dot{x} = f(x) \]

- \( x \in \mathbb{R} \)
- \( f \) does not depend on time
<table>
<thead>
<tr>
<th>Number of variables</th>
<th>N=1</th>
<th>N=2</th>
<th>N=3</th>
<th>N&gt;&gt;1</th>
<th>N=∞ (PDEs DDEs)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Linear</strong></td>
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<tr>
<td>RC circuit</td>
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<tr>
<td>Harmonic oscillator</td>
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<tr>
<td>dx(t)/dt = Ax(t) + Bu(t)</td>
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<td></td>
</tr>
<tr>
<td>x(t) = (x₁(t), ..., xₙ(t))</td>
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<tr>
<td>Logistic population grow</td>
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<td>Pendulum</td>
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<td>Forced oscillator</td>
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<tr>
<td>Lorentz model</td>
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</tbody>
</table>

**Nonlinear**

- Heat equation,
- Maxwell equations,
- Schrödinger equation,
- Navier-Stokes (turbulence)

"flow on the line"

PDEs=partial differential eqs.
DDEs=delay differential eqs.
Introduction to dynamical systems
Introduction to flows on the line
Solving equations with computer
Fixed points and linear stability
Feedback control: delay differential equations
Numerical integration

\[ \dot{x} = f(x) \]

- Euler method

\[
x(t_0 + \Delta t) = x_1 = x_0 + f(x_0)\Delta t
\]

\[
x_{n+1} = x_n + f(x_n)\Delta t
\]

\[
t_n = t_0 + n\Delta t
\]

- Euler second order

\[
\tilde{x}_{n+1} = x_n + f(x_n)\Delta t
\]

\[
x_{n+1} = x_n + \frac{1}{2}[f(x_n) + f(\tilde{x}_{n+1})]\Delta t.
\]
Fourth order (Runge-Kutta 1905)

\[
\begin{align*}
k_1 &= f(x_n) \Delta t \\
k_2 &= f(x_n + \frac{1}{2} k_1) \Delta t \\
k_3 &= f(x_n + \frac{1}{2} k_2) \Delta t \\
k_4 &= f(x_n + k_3) \Delta t.
\end{align*}
\]

\[x_{n+1} = x_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)\]

Problem if \( \Delta t \) is too small: round-off errors (computers have finite accuracy).
Table 12.1. *MATLAB’s ODE solvers.*

<table>
<thead>
<tr>
<th>Solver</th>
<th>Problem type</th>
<th>Type of algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>ode45</td>
<td>Nonstiff</td>
<td>Explicit Runge–Kutta pair, orders 4 and 5</td>
</tr>
<tr>
<td>ode23</td>
<td>Nonstiff</td>
<td>Explicit Runge–Kutta pair, orders 2 and 3</td>
</tr>
<tr>
<td>ode113</td>
<td>Nonstiff</td>
<td>Explicit linear multistep, orders 1 to 13</td>
</tr>
<tr>
<td>ode15s</td>
<td>Stiff</td>
<td>Implicit linear multistep, orders 1 to 5</td>
</tr>
<tr>
<td>ode23s</td>
<td>Stiff</td>
<td>Modified Rosenbrock pair (one-step), orders 2 and 3</td>
</tr>
<tr>
<td>ode23t</td>
<td>Mildly stiff</td>
<td>Trapezoidal rule (implicit), orders 2 and 3</td>
</tr>
<tr>
<td>ode23tb</td>
<td>Stiff</td>
<td>Implicit Runge–Kutta type algorithm, orders 2 and 3</td>
</tr>
</tbody>
</table>
Example 1

- **`quiver(x,y,u,v,scale)`**: plots arrows with components \((u,v)\) at the location \((x,y)\).

- The length of the arrows is \(scale\) times the norm of the \((u,v)\) vector.

To plot the blue arrows:

```matlab
%vector_field.m
n=15;
tpts = linspace(0,10,n);
ypts = linspace(0,2,n);
[t,y] = meshgrid(tpts,ypts);
pt = ones(size(y));
py = y.*(1-y);
quiver(t,y,pt,py,1);
xlim([0 10]), ylim([0 2])
```

\[
\dot{y} = y(1 - y)
\]
\[ \dot{y} = y(1 - y) \quad y(0) = 0.1 \]

To plot the solution (in red):

```matlab
\begin{verbatim}
  tspan = [0 10];
  yzero = 0.1;
  [t, y] = ode45(@myf, tspan, yzero);
  plot(t, y, 'r*--'); xlabel t; ylabel y(t)
\end{verbatim}
```

The solution is always tangent to the arrows.

Remember: HOLD to plot together the blue arrows & the trajectory.
\[ \dot{y} = -y - 5e^{-t} \sin 5t \quad y(0) = -0.5 \]

```matlab
n=15;
tpts = linspace(0,3,n);
ypts = linspace(-1.5,1.5,n);
[t,y] = meshgrid(tpts,ypts);
pt = ones(size(y));
py = -y-5*exp(-t).*sin(5*t);
quiver(t,y,pt,py,1);
xlim ([0 3.2]), ylim([-1.5 1.5])
```

```matlab
tspan = [0 3];
yzero = -0.5;
[t, y] = ode45(@myf,tspan,yzero);
plot(t,y,'kv--'); xlabel t; ylabel y(t)
```

```matlab
function yprime = myf(t,y)
yprime = -y -5*exp(-t)*sin(5*t);
```
General form of a call to Ode45

\[
[t, y] = \text{ode45}(@fun, \text{tspan}, \text{yzero}, \text{options}, p1, p2, \ldots);
\]

The optional trailing arguments \( p1, p2, \ldots \) represent problem parameters that, if provided, are passed on to the function \( \text{fun} \). The optional argument \( \text{options} \) is a structure that controls many features of the solver and can be set via the \( \text{odeset} \) function. In our next example we create a structure \( \text{options} \) by the assignment

\[
\text{options} = \text{odeset}('\text{AbsTol}', 1e-7, '\text{RelTol}', 1e-4);
\]

Passing this structure as an input argument to \( \text{ode45} \) causes the absolute and relative error tolerances to be set to \( 10^{-7} \) and \( 10^{-4} \), respectively. (The default values are \( 10^{-6} \) and \( 10^{-3} \); see \text{help odeset} for the precise meaning of the tolerances.) These
2.8.3  (Calibrating the Euler method) The goal of this problem is to test the Euler method on the initial value problem $\dot{x} = -x$, $x(0) = 1$.

a) Solve the problem analytically. What is the exact value of $x(1)$?

b) Using the Euler method with step size $\Delta t = 1$, estimate $x(1)$ numerically—call the result $\hat{x}(1)$. Then repeat, using $\Delta t = 10^{-n}$, for $n = 1, 2, 3, 4$.

c) Plot the error $E = |\hat{x}(1) - x(1)|$ as a function of $\Delta t$. Then plot $\ln E$ vs. $\ln \Delta t$. Explain the results.

2.8.4  Redo Exercise 2.8.3, using the improved Euler method.

2.8.5  Redo Exercise 2.8.3, using the Runge–Kutta method.
• Introduction to dynamical systems
• Introduction to flows on the line
• Solving equations with computer
• **Fixed points and linear stability**
• Feedback control: delay differential equations
Example

\[ \dot{x} = \sin x \]

Analytical Solution: 

\[ t = \ln \left| \frac{\csc x_0 + \cot x_0}{\csc x + \cot x} \right| \]

- Starting from \( x_0 = \frac{\pi}{4} \), what is the long-term behavior (what happens when \( t \to \infty \)?)

- And for any arbitrary condition \( x_0 \)?

- We look at the “phase portrait”: geometrically, picture of all possible trajectories (without solving the ODE analytically).

- Imagine: \( x \) is the position of an imaginary particle restricted to move in the line, and \( \frac{dx}{dt} \) is its velocity.
Imaginary particle moving in the horizontal axis

\[ \dot{x} = \sin x \]

Flow to the right when \( \dot{x} > 0 \)
Flow to the left when \( \dot{x} < 0 \)

\[ \dot{x} = 0 \quad \text{“Fixed points”} \]

Two types of FPs: stable & unstable
Fixed points

\[ \dot{x} = f(x) \quad f(x^*) = 0 \]

\[ x = x^* \text{ initially, then } x(t) = x^* \text{ for all time} \]

- Stable (attractor or sink): nearby trajectories are attracted \( \pi \) and \(-\pi\)

- Unstable: nearby trajectories are repelled \( 0 \) and \( \pm 2\pi \)

Fixed points = equilibrium solutions

\[ x = \sin x \]
Example 1

\[ \dot{x} = x^2 - 1 \]

- Find the fixed points and classify their stability

\[ f(x) = x^2 - 1 \]

\[ x^* = -1 \] is stable, and \( x^* = 1 \) is unstable
Example 2

\[-V_0 + R \dot{Q} + \frac{Q}{C} = 0\]

\[\dot{Q} = f(Q) = \frac{V_0}{R} - \frac{Q}{RC}\]
Example 3: population model for single species (e.g., bacteria)

- $N(t)$: size of the population of the species at time $t$

$$\frac{dN}{dt} = \text{births} - \text{deaths} + \text{migration}$$

- Simplest model (Thomas Malthus 1798): no migration, births and deaths are proportional to the size of the population

$$\frac{dN}{dt} = bN - dN \implies N(t) = N_0e^{(b-d)t}$$

Exponential growth!
More realistic model: logistic equation

- To account for limited food (Verhulst 1838):
  \[ \dot{N} = rN \left(1 - \frac{N}{K}\right) \]
  - If \( N > K \) the population decreases
  - If \( N < K \) the population increases

- \( K = \text{“carrying capacity”} \)

- The carrying capacity of a biological species in an environment is the maximum population size of the species that the environment can sustain indefinitely, given the food, habitat, water, etc.
How does a population approach the carrying capacity?

- Good model only for simple organisms that live in constant environments.
- Exponential or sigmoid approach.

\[
\dot{N} = rN \left(1 - \frac{N}{K}\right)
\]
Hyperbolic grow!
Technological advance
→ increase in the carrying capacity of land for people
→ demographic growth
→ more people
→ more potential inventors
→ acceleration of technological advance
→ accelerating growth of the carrying capacity…

Source: wikipedia
Linearization close to a fixed point

\[ \dot{x} = f(x) \quad f(x^*) = 0 \quad \eta(t) = x(t) - x^* \quad \eta = \text{tiny perturbation} \]

\[ \dot{\eta} = \frac{d}{dt}(x - x^*) = \dot{x} \]

\[ \dot{\eta} = \dot{x} = f(x) = f(x^* + \eta) \]

Taylor expansion

\[ f(x^* + \eta) = f(x^*) + \eta f'(x^*) + O(\eta^2) \]

The slope \( f'(x^*) \) at the fixed point determines the stability

- \( f'(x^*) > 0 \) the perturbation \( \eta \) grows exponentially
- \( f'(x^*) < 0 \) the perturbation \( \eta \) decays exponentially
- \( f'(x^*) = 0 \) Second-order terms can not be neglected and a nonlinear stability analysis is needed.

\[ \frac{1}{|f'(x^*)|} \] Characteristic time-scale

Bifurcation (more latter)
Existence and uniqueness

the solution to \( \dot{x} = x^{1/3} \) starting from \( x_0 = 0 \) is not unique.

- Problem: \( f'(0) \) infinite

- When the solution of \( dx/dt = f(x) \) with \( x(0) = x_0 \) exists and is unique?

- Short answer: if \( f(x) \) is “well behaved”, then a solution exists and is unique.

- “well behaved”?

- \( f(x) \) and \( f'(x) \) are both continuous on an interval of \( x \)-values and that \( x_0 \) is a point in the interval.

- Details: see Strogatz section 2.3.
Linear stability of the fixed points of \( \dot{x} = \sin x \)

- **Stable:** \( \pi \) and \(-\pi\)
- **Unstable:** 0, \( \pm 2\pi \)

\[
x^* = k\pi
\]
\[
f'(x^*) = \cos k\pi = \begin{cases} 1, & k \text{ even} \\ -1, & k \text{ odd.} \end{cases}
\]
Example 2

- Logistic equation

\[ \dot{N} = rN \left(1 - \frac{N}{K}\right) \]

\[ N^* = 0 \text{ and } N^* = K \]

\[ f'(0) = r \text{ and } f'(K) = -r \implies N^* = 0 \text{ is unstable} \]
\[ N^* = K \text{ is stable} \]

The two fixed points have the same characteristic time-scale:

\[ \frac{1}{|f'(N^*)|} = \frac{1}{r} \]
The population growth of the protozoan *Paramecium* in test tubes is a typical example (Figure 1.5). Under the conditions of the experiment, the population stopped growing when there were about 552 individuals per 0.5 ml. The time points show some scatter, which is caused both by the difficulty in accurately measuring population size (only a subsample of the population is counted) and by environmental variations over time and between replicate test tubes. A linear regression of the data $N'/N$ versus $N$ gives $r = 0.99$ and $K = 552$. 

\[ \dot{N} = rN \left( 1 - \frac{N}{K} \right) \]
Lack of oscillations

\[ \dot{x} = f(x) \]

General observation: only sigmoidal or exponential behavior, the approach is monotonic, **no oscillations**

Analogy:

\[ m\ddot{x} + b\dot{x} = F(x) \]

Strong damping (over damped limit)

To observe oscillations we need to keep the second derivative (weak damping).
Stability of the fixed point $x^*$ when $f'(x^*) = 0$?

(a) $\dot{x} = -x^3$  (b) $\dot{x} = x^3$  (c) $\dot{x} = x^2$  (d) $\dot{x} = 0$

In all these systems: $x^* = 0$ with $f'(x^*) = 0$

When $f'(x^*) = 0$ nothing can be concluded from the linearization but these plots allow to see what goes on.
Potentials

\[ \dot{x} = f(x) \quad f(x) = -\frac{dV}{dx} \quad \frac{dx}{dt} = -\frac{dV}{dx} \]

\[ \frac{dV}{dt} = \frac{dV}{dx} \frac{dx}{dt} \quad \frac{dV}{dt} = -\left(\frac{dV}{dx}\right)^2 \leq 0 \]

\( V(t) \) decreases along the trajectory.

- Example: \( \dot{x} = x - x^3 \)

\[ V = -\frac{1}{2} x^2 + \frac{1}{4} x^4 + C \]

Two fixed points: \( x=1 \) and \( x=-1 \) (Bistability).
Flows on the line = first-order ODE: $\frac{dx}{dt} = f(x)$

Fixed point solutions: $f(x^*) = 0$
   - stable if $f'(x^*) < 0$
   - unstable if $f'(x^*) > 0$
   - neutral (bifurcation point) if $f'(x^*) = 0$

There are no periodic solutions; the approach to a fixed point is monotonic (sigmoidal or exponential).
2.4.9 (Critical slowing down) In statistical mechanics, the phenomenon of "critical slowing down" is a signature of a second-order phase transition. At the transition, the system relaxes to equilibrium much more slowly than usual. Here's a mathematical version of the effect:

a) Obtain the analytical solution to $\dot{x} = -x^3$ for an arbitrary initial condition. Show that $x(t) \to 0$ as $t \to \infty$, but that the decay is not exponential. (You should find that the decay is a much slower algebraic function of $t$.)

b) To get some intuition about the slowness of the decay, make a numerically accurate plot of the solution for the initial condition $x_0 = 10$, for $0 \leq t \leq 10$. Then, on the same graph, plot the solution to $\dot{x} = -x$ for the same initial condition.

$$x(t) = \frac{x(0)}{\sqrt{1 + 2tx^2(0)}}$$
- Introduction to dynamical systems
- Introduction to flows on the line
- Solving equations with computer
- Fixed points and linear stability
- Feedback control: delay differential equations
Any system involving a feedback control will almost certainly involve time delays.

In a 2D system delayed feedback can reduce oscillations, but in a 1D system it can induce oscillations.

Example:

\[
\frac{dy}{dt} = ky(t - \tau), \quad y(t) = 1 \text{ when } -\tau \leq t < 0
\]

- Linear system
- Infinite-dimensional system
- Delay-induced oscillations.
In a single-species population, the incorporation of a delay allows to explain the oscillations, without the predatory interaction of other species.
It is important for the crane to move payloads rapidly and smoothly. If the gantry moves too fast the payload may start to sway, and it is possible for the crane operator to lose control of the payload.

Example: Container crane
Delayed feedback control

Figure 1.15: Container crane and ship (from H. Park and K.-S. Hong [181]).
Pendulum model for the crane, $y$ represents the angle

\[ y'' + \varepsilon y' + \sin(y) = -k \cos(y)(y(t - \tau) - y) \]

weakly damped oscillator Feedback control
(not first-order equation, without control payload oscillations are possible)

Reduction of payload oscillations: why delayed feedback works?

Near the equilibrium solution $y=0$: \[ y'' + \varepsilon y' + y = -k(y(t - \tau) - y) \]

Small delay: \[ y(t - \tau) \sim y - \tau y' \]

\[ y'' + (\varepsilon - k\tau)y' + y = 0 \]

The delay increases the damping. Therefore: the oscillations decay faster.
Small perturbation

- Figure 1.16: The values of the fixed parameters are $\tau = 12$, $\varepsilon = 0.1$, and $k = -0.15$. (a): $y'(0) = 0$ and $y = 1$ ($-\tau < t < 0$); (b): $y'(0) = 0$ and $y = 1.5$ ($-\tau < t < 0$).

Large perturbation
Example: Car following model

\[ x''_{n+1}(t + \tau) = \alpha(x'_n - x'_{n+1}) \]

can be used for determining the location and speed of the following car (at \( x = x_{n+1} \)) given the speed pattern of the leading vehicle (at \( x = x_n \)). If a driver reacts too strongly (large value of \( \alpha \) representing excessive braking) or too late (long reaction time \( \tau \)), the spacing between vehicles may become unstable (i.e., we note damped oscillations in the spacing between vehicles).
Typical solution for two cars

The lead vehicle reduces its speed of 80 km/h to 60 km/h and then accelerates back to its original speed. The initial spacing between vehicles is 10 m.

\[ x''_{n+1}(t + \tau) = \alpha(x'_n - x'_{n+1}) \]

\[ \alpha = 0.5 \text{ s}^{-1} \text{ and } \tau = 1 \text{ s} \]

- Speed of the two cars
- Distance between the two cars
A sober driver needs about 1 s in order to start breaking in view of an obstacle. With 0.5 g/l alcohol in blood (2 glasses of wine), this reaction time is estimated to be about 1.5 s. \(\Rightarrow\) oscillations near the stable equilibrium increase.
Example 1: Delayed logistic equation

\[ \frac{dy}{dt} = \lambda y(1 - y(t - 1)) \]

```matlab
function solve_delay1
    tau = 1;
    ic = [0.5];
    tspan = [0 100];
    h = 1.8;
    sol = dde23(@(t,y,Z) h*y(1).*(1-Z(1)), tau, ic, tspan);
    plot(sol.x, sol.y(1,:), 'r-')
end
```

```matlab
function v = f(t, y, Z)
    v = [h*y(1).* (1-Z(1))];
end
```
Example 2: Prey (x) and predator (y) model

\[
\frac{dx}{dt} = x(t) \left\{ 2 \left[ 1 - \frac{x(t)}{50} \right] - \frac{y(t)}{x(t) + 40} \right\} - 10,
\]

\[
\frac{dy}{dt} = y(t) \left[ -3 + \frac{6x(t - \tau)}{x(t - \tau) + 40} \right].
\]

```matlab
function solve_delay2
tau=9;
ic = [35;10];
tspan = [0 250];
h = 10;
sol = dde23(@f,tau,ic,tspan);
plot(sol.x,sol.y(1,:),'r-',sol.x,sol.y(2,:),'b--')
end
```

```matlab
function v=f(t,y,Z)
v = [y(1)*(2*(1-y(1)/50)-y(2)/(y(1)+40))-h*y(2)*(-3+6*Z(1)/(Z(1)+40))];
end
```
Steven H. Strogatz: *Nonlinear dynamics and chaos, with applications to physics, biology, chemistry and engineering.* First or second ed., Chapters 1 and 2
