VII Ciclo de Mini-cursos e Palestras

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O Método de Öpik

Notas de aula

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1. Sphere of action and Hill's radius, R_H

Consider the acceleration due to the Sun at a distance r: $\alpha = GM_\odot/r^2$

A small dr generates: $d\alpha = -2GM_\odot \frac{1}{r^3} dr$

Consider a satellite orbiting a planet at a distance ρ. Taking $dr = \rho$ the difference in the acceleration between planet located at a mean heliocentric distance a and the satellite is $d\alpha = 2GM_\odot \frac{1}{a^3}\rho$. The aceleration due to the planet is Gm/ρ^2 and when both are comparable the satellite lost its planetocentric regime and that occurs for a limit value ρ_L:

$$\rho_L = a\Big(\frac{m}{2M_\odot}\Big)^{1/3}$$

Outside this sphere a planet cannot retain a satellite. A more standard parameter is the Hill's radius (derived from the R3BP):

$$R_H = a\Big(\frac{m}{3M_\odot}\Big)^{1/3}$$

Then

$$\frac{SolarTide}{PlanetaryAcceleration} \sim \frac{2}{3}\Big(\frac{\rho}{R_H}\Big)^3$$

2. Hyperbolic encounters and impact parameter, σ

Once an asteroid is well inside the Hill's radius (for example when $r < R_H/4$) of a planet we can neglect the perturbations by the Sun and consider the hyperbolic planetocentric trajectory of the asteroid. The equation of energy is

$$\frac{1}{2}v^2 - \frac{Gm}{r} = -\frac{Gm}{2a}$$

The planetocentric velocity is $v_\infty = |\vec{V}_a - \vec{V}_p|$ which can be considered at infinity because the term Gm/r can be neglected in front of $v^2/2$. This is valid if $v^2 >> 2Gm/r$ at a planetocentric distance where the sun's attraction can be neglected. For example at $r = R_H/4$ the condition becomes

$$v(km/s) >> 100(\frac{m}{M_\odot})^{2/3}$$

For example, for the Earth we have the condition $v >> 0.02km/s$ which is almost always satisfied taking into account that the mean velocity of encounter for the Earth with asteroids is 2 to 3 orders of magnitude greater.

Then when $r \sim R_H/4$ is reasonable to assume a) the problem is planetocentric and b) the asteroid is "at infinity" with respect to the planet.

The semimajor axis of the planetocentric hyperbola is

$a = -\frac{Gm}{v_\infty^2}$

and the angular momentum is

$h = \sigma \cdot v_\infty = q \cdot v_q$

being σ the **impact parameter** which can be related to the **pericentric distance** of the trajectory q:

$\sigma = q\sqrt{1 - 2a/q} = q\sqrt{1 + \frac{2Gm}{v_\infty^2 q}}$

A collision with the planet occurs when $q \leq R_{planet}$. We can define the σ corresponding to collision

$$\sigma_c = R_p\sqrt{1 + \frac{2Gm}{v_\infty^2 R_p}}$$

for $\sigma < \sigma_c$ a collision is certain.

The angular deflection of the planetocentric velocity $\vec{v}$ is γ (see figure 1):

$\tan\frac{\gamma}{2} = \frac{Gm}{\sigma v_\infty^2}$

or

$\sin\frac{\gamma}{2} = (1 + \sigma v_\infty^2/Gm)^{-1}$

these formulae can be deduced from the conic when the true anomaly tends to infinity: $\cos f_\infty = -1/e = -\sin(\gamma/2)$.

Total randomization of $\vec{v}$ is attained if encounters are such that $\gamma \geq 90^o$ which means $\sigma \leq Gm/v_\infty^2$. In some cases randomization would be attained only for such σ that a collision with the planet is inevitable.

(see figure 2)

3. Restricted Three-Body Problem, Jacobi's constant C

Consider a planet revolving a star with circular orbit of radius a (see figure 3). We redefine units of mass, length and time such that:

$1 - m$: mass of star

m: mass planet

the mean motion is $n^2 = \mu/a^3$

the constant for two body problem is: $G(m_1 + m_2) = \mu = k^2((1-m)+m) = k^2$

taking unit of length equal to a and taking unit of time such as to make $k = 1$ then

$n = 1 = \frac{2\pi}{P}$ then using these units the orbital period of the planet is $P = 2\pi$. The linear velocity of the planet around the star is $V_p = na = 1$.

We define the system $(\hat{x}, \hat{y}, \hat{z})$ which rotates with the planet around the baricenter of the system with angular velocity $\vec{\omega} = n\hat{z} = 1\hat{z}$.

Consider a particle located in $\vec{r} = (x, y, z)$. We can demonstrate the Jacobi's integral of motion of the particle where v is the particle's velocity in the rotating frame:

$$v^2 = x^2 + y^2 + \frac{2(1-m)}{r_1} + \frac{2m}{r_2} - C$$

being r_i the distance to mass i and C is a constant.

Demonstration:

The velocity in the inertial frame $\vec{V}$ and the one in the rotating frame $\dot{\vec{r}}$ are related by

$\vec{V} = \dot{\vec{r}} + \vec{\omega} \wedge \vec{r}$

The inertial acceleration is $\vec{\alpha} = -\nabla\mathbb{V}$ where $\mathbb{V}(\vec{r}) = -(1-m)/r_1 - m/r_2$ is the gravitational potential generated by the two masses.

The rotating system rotates with $\vec{\omega} = \hat{z}$ then the relationship between inertial acceleration $\vec{\alpha}$ and the acceleration relative to the rotating system $\ddot{\vec{r}}$ is

$\vec{\alpha} = \ddot{\vec{r}} + 2\hat{z} \wedge \dot{\vec{r}} + \hat{z} \wedge (\hat{z} \wedge \vec{r})$

but $\vec{r} = z\hat{z} + \vec{\rho}$ being $\vec{\rho} = (x, y, 0)$

then

$\vec{\alpha} = \ddot{\vec{r}} + 2\hat{z} \wedge \dot{\vec{r}} - \vec{\rho}$

multiply by $\dot{\vec{r}}$:

$\vec{\alpha} \cdot \dot{\vec{r}} = \left[\ddot{\vec{r}} \cdot \dot{\vec{r}} - \vec{\rho} \cdot \dot{\vec{\rho}}\right]$

then

$\vec{\alpha} \cdot d\vec{r} = -\nabla\mathbb{V} d\vec{r} = \left[\ddot{\vec{r}} \cdot \dot{\vec{r}} - \vec{\rho} \cdot \dot{\vec{\rho}}\right] dt$

integrating

$$-2\mathbb{V}(\vec{r}) = \dot{\vec{r}}^2 - (x^2 + y^2) + C$$

or

$$v^2 = x^2 + y^2 - 2\mathbb{V}(\vec{r}) - C$$

then, the particle's velocity in the rotating frame becomes

$$v^2 = x^2 + y^2 + \frac{2(1-m)}{r_1} + \frac{2m}{r_2} - C$$

C is a constant in the R3BP. If planet's eccentricity is different from zero C will oscillate around a mean value.

4. Tisserand parameter, T

The particle has some orbital elements (a, e, i) and we will make to appear them in Jacobi's integral. We need to express position and velocity in the rotating frame $(\vec{r}, \vec{v})$ as function of position and velocity $\vec{V}$ in the inertial frame.

We have

$\vec{V} = \dot{\vec{r}} + \vec{\omega} \wedge \vec{r} = \dot{\vec{r}} + \hat{z} \wedge \vec{\rho}$

Then

$\dot{\vec{r}} = \vec{V} - \hat{z} \wedge \vec{\rho}$

squaring

$v^2 = \vec{V}^2 - 2\vec{V} \cdot (\hat{z} \wedge \vec{\rho}) + \rho^2$

rearranging

$v^2 = \vec{V}^2 - 2\hat{z} \cdot (\vec{\rho} \wedge \vec{V}) + \rho^2$

$v^2 = \vec{V}^2 - 2\hat{z} \cdot (\vec{r} \wedge \vec{V}) + x^2 + y^2$

$\vec{V}^2 - 2\hat{z} \cdot (\vec{r} \wedge \vec{V}) = v^2 - x^2 - y^2 = \frac{2(1-m)}{r_1} + \frac{2m}{r_2} - C$

(in a numerical integration it is easier to calculate C using the inertial frame than the rotating one)

According to the two body problem baricenter-particle:

$V^2 = 2/r - 1/a$ and $\hat{z} \cdot (\vec{r} \wedge \vec{V}) = \hat{z} \cdot \vec{h} = \sqrt{a(1-e^2)} \cos i$

then

$\frac{2}{r} - \frac{1}{a} - 2\sqrt{a(1-e^2)}\cos i = \frac{2(1-m)}{r_1} + \frac{2m}{r_2} - C$

The orbital elements (a, e, i) are referred to the baricenter of the system Star+planet and the inclination is measured with respect to the orbital plane $\hat{x}\hat{y}$ of the planet. In the case of the solar system $m < 10^{-3}$ so it is possible to assume that (a, e, i) are heliocentric.

If the particle **is not very close to the Sun** we have $r \simeq r_1$ then

$\frac{1}{a} + 2\sqrt{a(1-e^2)}\cos i = 2m[\frac{1}{r_1} - \frac{1}{r_2}] + C$

If the particle is far from the sun and from the planet and taking into account that $m < 10^{-3}$ we obtain

$$C \simeq \frac{1}{a} + 2\sqrt{a(1-e^2)}\cos i = T$$

T is known as the Tisserand parameter. In the R3BP C is constant and T presents some departures if the orbital elements are determined when the conditions above are not satisfied (near the sun or the planet). T should be considered as a simple form of calculating C.

For elliptic orbits it is possible to express $T(q, Q, i)$ where q, Q are perihelion and aphelion:

$$T = \frac{2}{q+Q} + 2\sqrt{2qQ/(q+Q)}\cos i$$

This is a useful formula when analyzing regions where encounters are possible $(q < 1, Q > 1)$.

(see figures 4-7)

5. Öpik begins. The encounter velocity, U

Suppose the particle is near the planet ($r_1 \simeq 1$ and $x^2 + y^2 \simeq 1$) but far enough that we can neglect its gravitational attraction ($r \sim R_H$) so the particle is "at infinity" ($m/r_2 \simeq 0$). Then from Jacobi's integral:

$$v_\infty^2 \simeq 1 + 2 + 0 - T$$

then, under the hypothesis above, the planetocentric velocity "at infinity" of the particle is

$$v_\infty \simeq \sqrt{3 - T} = U$$

U is the encounter velocity with the planet **before** the gravitational attraction is felt by the particle (that means "at infinity"). U is determined by T which is constant, so U is also constant.

The orbital elements (a, e, i) can evolve but T and U remain constant, only the orientation of $\vec{U}$ is modified (U rotates γ after the encounter).

It follows that when $T > 3$ encounters cannot exist. When $T < 3$ they could exist but they are not guaranteed. For example: $a = 2, e = 0, i = 90^o$ implies $T = 0.5$ but the particle never approaches the planet.

If $U \sim 0$ the planetocentric orbit is quasi-parabolic and a temporary capture by the planet is possible. Then, objects with $T \sim 3$ can experience temporary captures by the planet.

The greatest heliocentric velocity the particle can get after the encounter is $V_p + U = 1 + U$. The escape velocity from the system is $\sqrt{2}$, so if $U \geq \sqrt{2} - 1$ the particle eventually can escape from the solar system and conversely if $U < \sqrt{2} - 1$ the particle will never left the solar system by this mechanism. Note that only prograde orbits have $U < 1$.

The final heliocentric velocity is a vectorial sum (see figure 8):

$\vec{V} = \vec{V_p} + \vec{U'}$ or $V^2 = 1 + U^2 + 2U \cos\theta$

being θ the angle between $\vec{V_p}$ and $\vec{U'}$. If $U > \sqrt{2} - 1$ there exists some θ_∞ so that for $\theta \leq \theta_\infty$ the corresponding V is greater than the ejection velocity. This situation occurs for

$\cos\theta_\infty = \frac{1-U^2}{2U}$

If we can assume that $\vec{U'}$ is randomized (deflection γ is so great that θ can get all values from 0 to π) then the **probability of ejection per encounter** is equal to the probability $P(\theta \leq \theta_\infty)$ and this is equal to the solid angle subtended by θ_∞ over 4π which is equal to

$$P_\infty = P(\theta \leq \theta_\infty) = \frac{1}{2}(1 - \cos\theta_\infty) = \frac{U^2 + 2U - 1}{4U} \qquad (U > \sqrt{2} - 1, \gamma > 90^o)$$

Conversely, a comet in an hyperbolic heliocentric orbit has a probability of being captured after an encounter and is equal to $1 - P_\infty$. These results are only valid for encounters satisfying the conditions $(U > \sqrt{2} - 1, \gamma > 90^o)$. These are very strong conditions, for example, a particle encountering the Earth never satisfies $\gamma > 90^o$ with $\sigma > \sigma_c$. So, the P_∞ should be weighted with the probability $P(\gamma \geq 90^o)$ which is very low. Weidenschilling (1975) recalculate this issue obtaining more realistic values for the ejection probability (see figures 9-10). In section 9 we explain how to calculate this issue.

6. Geometry of encounters

The velocity of encounter U form an angle θ with the planet's heliocentric velocity ($\vec{V_p} = \hat{y}$) and is rotated an azimuthal angle ϕ around $\hat{y}$ (see figure 11). Then:

$U_x = U \sin\theta \sin\phi$

$U_y = U \cos\theta$

$U_z = U \sin\theta \cos\phi$

Assuming the asteroid is encountering the planet: $r = 1$ and $V^2 = 2 - 1/a$. The "angular momentum" is $\sqrt{a(1-e^2)} = rV_t$ where V_t is the transverse velocity. In consequence the radial velocity evaluated at $r = 1$ is

$$V_r^2 = V^2 - V_t^2 = 2 - 1/a - a(1-e^2)$$

The encounter with the planet occurs at the line of the nodes of the asteroid's orbit then (see figure 12):

$V_y = V_t \cos i$

$V_z = V_t \sin i$

$V_x = V_r$

Then the relative velocity $\vec{U} = \vec{V} - \vec{V}_p = (V_x, V_y - 1, V_z)$ has components

$U_x = \pm\sqrt{2 - 1/a - a(1-e^2)}$

minus sign is for encounters at pre perihelion passage ($\dot{r} < 0$)

$U_y = \sqrt{a(1-e^2)} \cos i - 1$

$U_z = \pm\sqrt{a(1-e^2)} \sin i$

minus sign is for encounters at the descending node of the asteroid's orbit ($\dot{z} < 0$)

Conversely

$a = \frac{1}{1-U^2-2U_y}$

$e = \sqrt{U^4 + 4U_y^2 + U_x^2(1-U^2-2U_y) + 4U^2U_y}$

$i = \arctan\frac{U_z}{1+U_y}$

or also

$\sin^2 i = \frac{U_z^2}{U_z^2+(1+U_y)^2}$

If we define the heliocentric "energy" of the particle as $x = 1/a$ then we have

$x = 1/a = (1 - U^2 - 2U\cos\theta)$

and the variation in the energy due to the encounter is

$\Delta x = 1/a' - 1/a = 2U(\cos\theta - \cos\theta')$

Maximum variations in energy are (see figure 13): $\Delta x = 2U(1 - \cos\gamma(\sigma, U))$

7. Probability of encounters, p

Consider an asteroid in an heliocentric orbit with $q < 1$ (see figure 14). It crosses two times the sphere of radius $r = 1$. We will find an expression for the probability of encounter with the planet inside an impact parameter σ. We know that at $r = 1$ the radial velocity of the asteroid is $V_r = U_x$. Then the time spent in the spherical shell of thickness dr and radius 1 is:

$$dt = 2dr/U_x$$

The probability **per revolution** to find the asteroid in the shell is:

$$dN = dt/P$$

being P the orbital period. The asteroid only can be found inside a band which is bounded by two parallels of latitude $\pm i$ (see figure 15) and whose volume is

$$dV = 4\pi \sin i dr$$

Assuming Ω and ω circulating, the average density of the asteroid in the volume is then

$$\varrho_i = dN/dV$$

Near the reference plane (where the planet revolves) the density is something small because of the higher rate of variation of the latitude near the node of the asteroid. An analogy with the Sun can help: the variation rate of the altitude of the Sun is higher at sunrise or sunset than at noon. Taking this into account it is possible to show that the mean density near the reference plane is

$$\varrho_o = \frac{2}{\pi}\varrho_i$$

The planet is moving with respect to the asteroid with velocity U and it defines a small cylinder with radius σ per unit of time given by

$dV/dt = \pi\sigma^2 U$

then the number of encounters per revolution is

$$dV/dt \cdot \varrho_o \cdot P$$

substituting

$$\pi\sigma^2 U \cdot \frac{2}{\pi} dN/dV \cdot P = \pi\sigma^2 U \cdot \frac{2}{\pi} \frac{2dr}{|U_x|P} \frac{1}{4\pi \sin i dr} \cdot P$$

then, the **probability of an encounter with impact parameter $\leq \sigma$ per revolution** is

$$p(\sigma) = \frac{\sigma^2 U}{\pi \sin i |U_x|}$$

This is the famous formula given by Öpik (1951), valid for $\sigma < R_H$ where the two body scheme can be applied.

The mean number of orbital revolutions between these encounters is

$\nu = 1/p$

and the time in years between encounters (P is the orbital period)

$\tau = \nu P = a^{1.5}/p$

where a is in astronomical units.

The function p is a cumulative probability function. If we are interested in the probability of having encounters with impact parameter between σ and $\sigma + \Delta\sigma$:

$p(\sigma + \Delta\sigma) - p(\sigma) = \Delta p = \frac{dp}{d\sigma}\Delta\sigma$

then

$$p(\sigma = \sigma^*) = \frac{2\sigma^* U}{\pi \sin i |U_x|} \qquad \text{(for } \sigma^* < R_H\text{)}$$

Taking into account the probability density function above we can calculate the mean value $< \sigma >$ of the impact parameter for all possible values from 0 to R_H (limit of validity of two body approximation):

$$< \sigma >= \frac{2}{3} R_H$$

The function p may vary between encounters because of variations of i and $|U_x|$ but if we have a population of N objects with similar orbits given by a distribution of orbital elements which can be considered in a steady state, an average probability of the encounter per revolution can be devised:

$\bar{p} = \Sigma p_i / N$

If p represents the probability of elimination by collision with some planet or by ejection from the population, the **average lifetime** of the particles in years can be estimated by

$\bar{\tau} =< a >^{1.5} / \bar{p}$

where $< a >$ is the mean semimajor axis of the population.

Annually will be eliminated $N/\bar{\tau}$, then the population will evolve as

$$dN = -(N/\bar{\tau})dt$$

and the fraction of surviving objects after t years will be:

$N/N_o = e^{-t/\bar{\tau}}$

This exponential decay was critically revised by several authors (see figures 16 and 18). Numerical integrations in general show that the population decay exponentially at the very beginning and then a power law decay follows ($N/N_o \propto t^{\alpha}$).

Example. Consider a NEO with $(a, e, i) = (2, 0.7, 10^o)$ and calculate the collision probability with Earth. We obtain $T = 2.489, U = 0.715, U_x = 0.69$, then $p(\sigma_c) = 1.9 \cdot \sigma_c^2$ and taking into account $\sigma_c = 4.9 \times 10^{-5}$ we obtain $p_c = 4.5 \times 10^{-9}$ which gives $\bar{\tau} = 628$ millon years.

8. Omnidirectional encounter probabilities

After N encounters the mean total deflection Γ of the relative velocity $\vec{U}$ can be estimated from

$$\Gamma^2 = \Sigma\gamma_i^2 = N\bar{\gamma}^2$$

where $\bar{\gamma}$ is a mean value generated by the mean encounter parameter $< \sigma >$. We can adopt $\Gamma = \pi/2$ for full randomization, then the mean number of passages with $\sigma < R_H$ necessaries for full randomization of γ is

$$N = \pi^2/(4\bar{\gamma}^2)$$

For the example given above we have $< \sigma >= 6.7 \times 10^{-3}$ and $\bar{\gamma} = 1.8 \times 10^{-3}$ radians. Then full randomization is acquired after $N \sim 800000$ encounters with $\sigma < R_H$. N is not the number of revolutions just the number of encounters with $\sigma < R_H$, we need to consider the probability $p(R_H) = 0.00019$ and the time between these types of encounters $\bar{\tau} = 14900$ years. Then, to acquire full randomization it is necessary to wait for $t \sim 11600$ millon years. That means it is more probable a collision with the Earth before full randomization.

For small U we have large values of γ and N is relatively small, then we can assume that the vector $\vec{U}$ is randomized so we can assume **equipartition**

$$< U_x^2 >=< U_y^2 >=< U_z^2 >= U^2/3$$

and taking into account $< U_y >= 0$ we can estimate the mean orbital elements for a population having omnidirectional encounters with the planet:

$$< e^2 >= \frac{5}{3}U^2(1 + \frac{2}{5}U^2)$$

$$< \sin^2 i >= \frac{U^2}{3 + 2U^2}$$

then we can substitute $U/|U_x|$ by its average value $\sqrt{3}$ and $\sin i$ by $\frac{U}{\sqrt{3+2U^2}}$ obtaining the encounter probability under the condition of randomization

$$p(\sigma) = \frac{3\sqrt{1 + \frac{2}{3}U^2}}{\pi U}\sigma^2 \qquad\qquad (\vec{U} \quad \text{randomized})$$

9. Modeling the orbital evolution

Arnold (1965) devised a Monte Carlo method (reformulated by several authors) to simulate the orbital evolution of asteroids encountering the planets using the probability function obtained by Öpik. Consider the spherical triangle whose sides are given by θ, γ, θ' (see figure 17):

$\cos\theta' = \cos\theta\cos\gamma + \sin\theta\sin\gamma\cos\psi$

defining

$\chi = \phi - \phi'$

we can write

$\sin\chi = \sin\psi\sin\gamma/\sin\theta'$

$\cos\chi = (\cos\gamma\sin\theta - \sin\gamma\cos\theta\cos\psi)/\sin\theta'$

Then we can simulate the evolution as follows:

We have an asteroid with (a, e, i). Check that $q < 1$ and $Q > 1$.

Calculate U and (U_x, U_y, U_z) taking aleatory sign for U_x, U_z.

Calculate (θ, ϕ).

From a probability distribution function proportional to σ we choose the impact parameter at random ($0 < \sigma < R_H$)....(do you know how to do this? .... change to a variable y that has uniform distribution then $\sigma d\sigma = dy$ integrating: $\sigma = \sqrt{2y}$, so choose at random with equal probability y and calculate σ). This is equivalent to choose at random σ^2 from a uniform probability function between $0 < \sigma^2 < R_H^2$.

If $\sigma < \sigma_c$ the particle impact the planet. If not, calculate the corresponding deflection angle γ.

Compute the mean time τ for this event and compute the time as $t' = t + \tau$.

From a uniform probability distribution choose the azimuthal angle ψ at random ($0 < \psi < 2\pi$).

Calculate $\theta'(\theta, \gamma, \psi)$.

Calculate $\chi(\theta, \gamma, \psi, \theta')$.

Obtain $\phi' = \phi - \chi$.

Calculate the new (U'_x, U'_y, U'_z)

Calculate the new (a, e, i). If $a < 0$ the particle is ejected.

Choose a new set (σ, ψ).

If the asteroid's orbit intersects various planets, the modeling must include the encounters with all them.

Using the scheme above it is possible to numerically calculate the probability that the asteroid experience some variation in its orbital elements. For example, if we want to know the probability of having a certain Δx in its energy we choose at random with uniform distribution several points in the space ($\sigma_c^2 \le \sigma^2 \le R_H^2, 0 \le \psi \le 2\pi$) and calculate the corresponding Δx. If that value is the one we were looking for we increment a unit in a counter. After N points in the space (σ^2, ψ) we will succeed in N_s points. For $N \to \infty$ we can obtain the probability

$$p(\Delta x) = \frac{N_s}{N} \cdot p(R_H)$$

where $p(R_H)$ is the probability of an encounter with impact parameter below R_H calculated with the Öpik's formula.

10. Theory of diffusion

Previous to Öpik work some authors (Van Woerkom 1948, Oort 1950) analyzed the evolution of the orbital "energy" $x = 1/a$ of the comets as a diffusion problem.

If $< \sigma_x^2 >$ is the mean squared energy change per perihelion passage, the total mean change in energy, Δx, after N passages verify

$$< \Delta x >^2 = N < \sigma_x^2 >$$

The number of passages necessaries to make $\Delta x = x$ is $N = x^2/ < \sigma_x^2 >$ and the number of years required is

$$t_D = N x^{-3/2} = x^{1/2}/ < \sigma_x^2 >$$

this is the **energy diffusion time**.

From Yabushita (1980) it is possible to show that the **median lifetime** (when half of the population is ejected) is $t_{med} \sim 4.77 t_D$. A comparison of different models applied to simulate the evolution of Chiron is given at figure 18.

11. The problem of the asymmetries

Everhart (1969) numerically found that parabolic comets encountering Jupiter have not equal probability of increasing or decreasing its orbital energy $x = 1/a$. That was known as the problem of the asymmetries in the distribution of the Δx due to the planetary perturbations. Considering the Öpik formulation Carusi et al. (1990) explained the asymmetries as a natural consequence of the outcomes of the encounters. For example, a parabolic comet with $q = 0.1 a_J$ and $i = 27^o$ has $T = 0.797$ and $U = 1.484$ with respect to Jupiter. The maximum allowed changes in energy are $\Delta x = 1 - U^2 \pm 2U$ which corresponds to $\Delta x = -4.17$ and $\Delta x = +1.77$ so is reasonable that for this example negative changes Δx are more probable than positive changes, that means ejection is more probable than capture (see figure 19).

12. Some conclusions

The method as originally presented by Öpik in general overestimates the mean lifetime of the populations because it does not take into account the strong changes in e due to the resonances (see figure 20).

It cannot be applied to populations with very small U (see figure 21).

Collision probabilities predictions are consistent with numerical integrations (except collisions with Sun).

It can be generalized to eccentric and inclined planetary orbits.

It gives order of magnitude valid results and it is very fast.

Numerical integrations in general show that the population decay exponentially at the beginning and then a power law decay follow ($N/N_o \propto t^\alpha$).

Recommended bibliography

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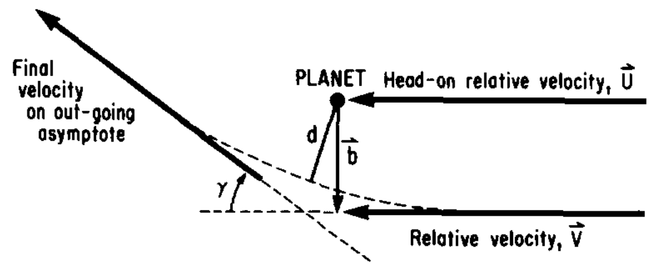

FIG. 2. The Öpik method. Approach velocity **V** (for impact parameter b) is assumed equal to **U**, the velocity for a head-on collision ($b = 0$) with the planet. **V** is taken as the asymptotic approach velocity for a two-body hyperbolic encounter. Closest approach d and rotation angle γ are based on the two-body encounter (see Eqs. (1) and (2)). The final velocity has the direction of the outgoing asymptote and the magnitude V.

Fig. 1.— Asteroid encountering a planet. Greenberg 1982.

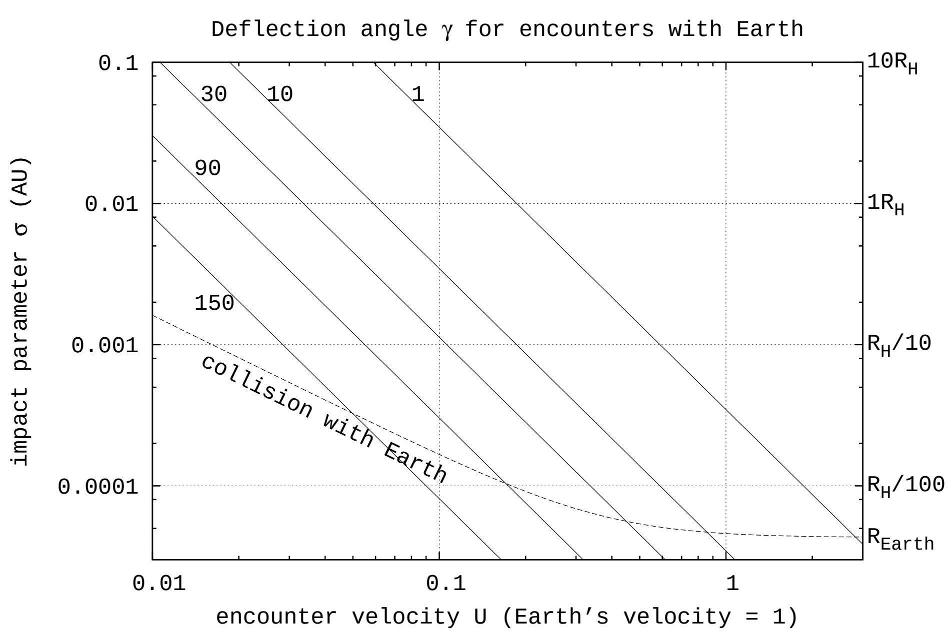

Fig. 2.— Deflection angle $\gamma(\sigma, U)$.

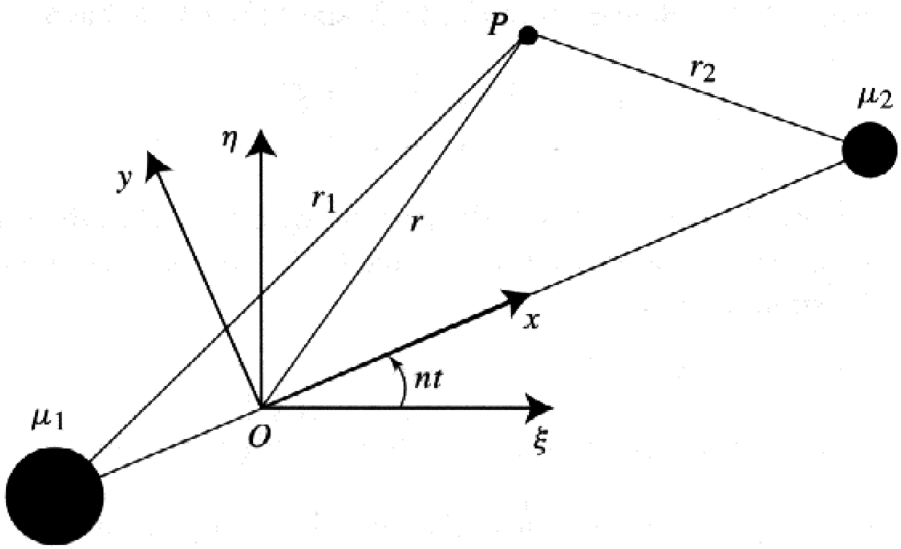

Fig. 3.1. A planar view of the relationship between the sidereal coordinates (ξ, η, ζ) and the synodic coordinates (x, y, z) of the particle at the point P. The origin O is located at the centre of mass of the two bodies. The ζ and z axes coincide with the axis of rotation and the arrow indicates the direction of positive rotation.

Fig. 3.— R3BP, rotating and inertial frames. Murray and Dermott 1999.

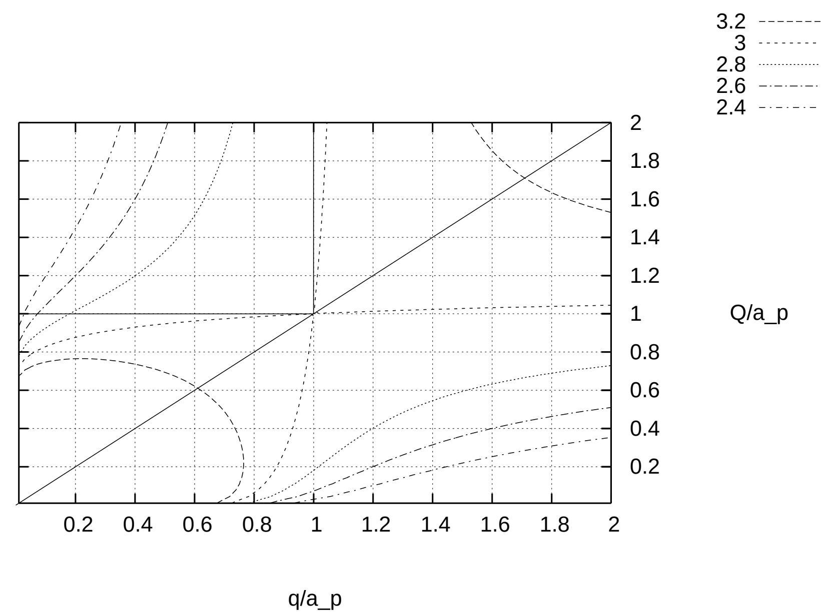

Fig. 4.— Tisserand parameter $T(q, Q, i = 0)$. The region below the diagonal is not real.

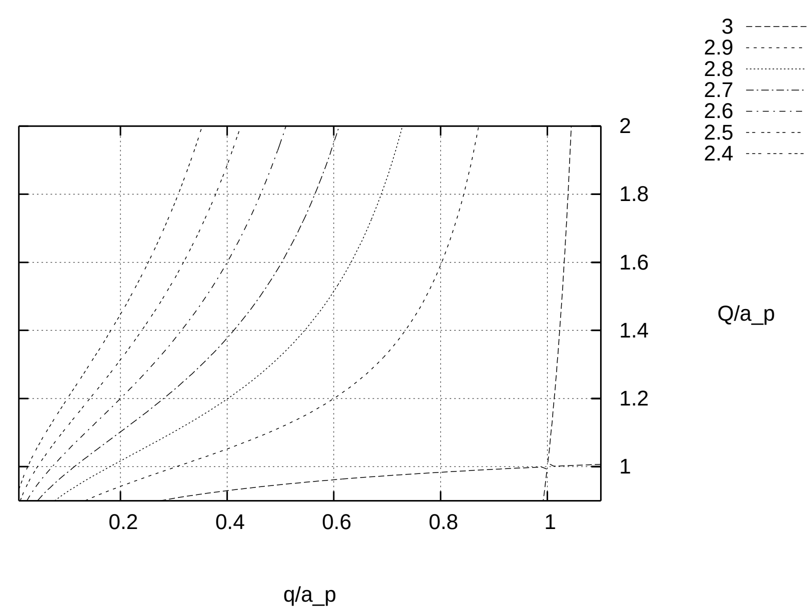

Fig. 5.— Tisserand parameter $T(q, Q, i = 0)$. The region where encounters are possible.

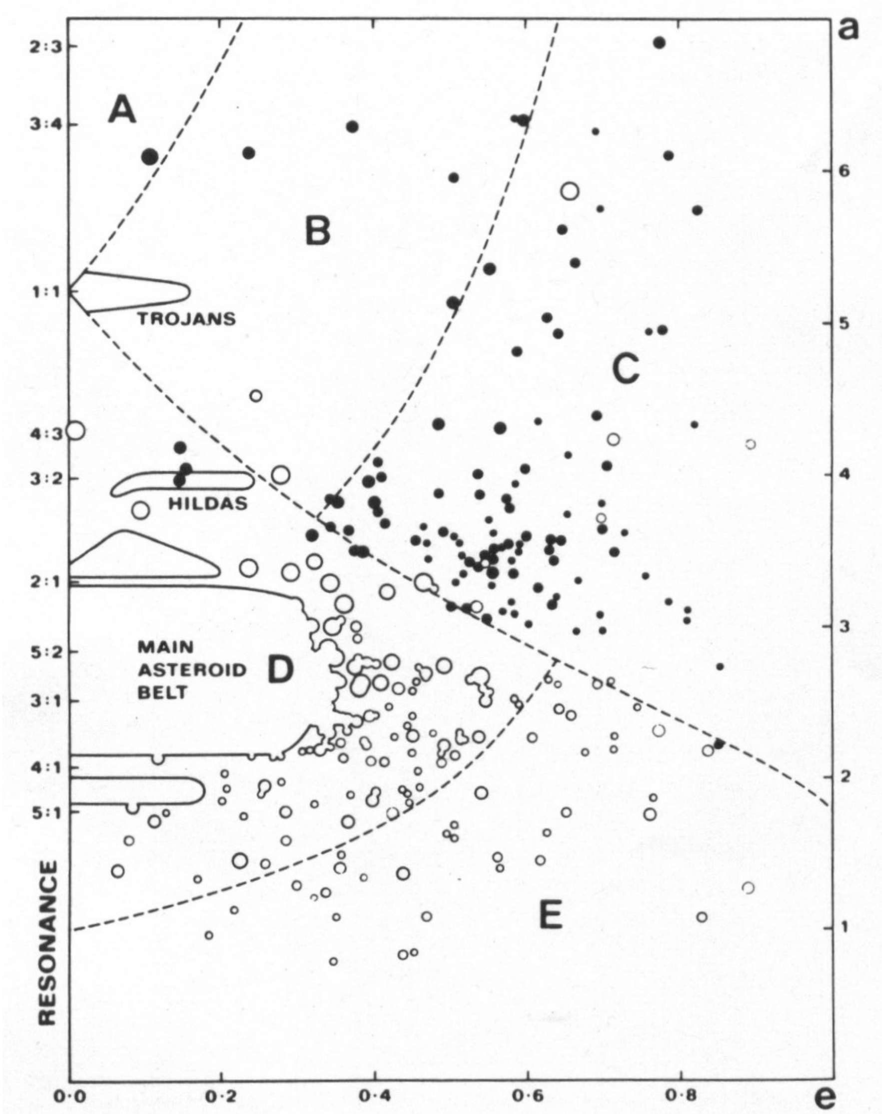

Fig. 1. Short-period comets (solid circles) and asteroids (open circles) plotted on a scatter diagram of semimajor axis vs eccentricity (Kresák 1985). Increasing circle size indicates estimated size of the objects: diameter < 1 km or lost, 1 to 3 km, 3 to 10 km, 10 to 30 km and > 30 km. Different regions identified within the diagram are: (A) transjovian region, (B) Jupiter domain of weak cometary activity, (C) Jupiter domain of strong cometary activity, (D) minor planets region, and (E) Apollo-Aten region. The dashed line going from upper left to lower right corresponds to a Tisserand invariant of 3.0, the usual dividing line between comets and asteroids. However, note the several asteroids above the line in the cometary region C; the figure has been modified to include seven new asteroids in or near region C discovered since Kresák's (1985) work was published.

Fig. 6.— Kresak's diagram. Regions B and C corresponds to $T < 3$ and regions A, D and E to $T > 3$. Asteroids II.

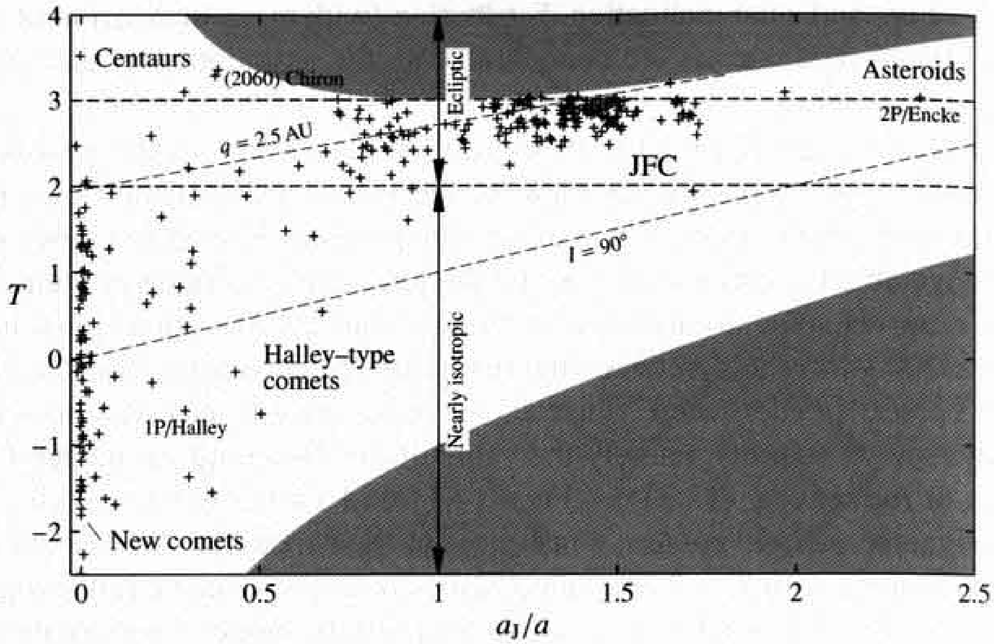

Figure 14.12. Position of known comets in the plane of the Tisserand parameter T and the ratio a_J/a between the semimajor axes of Jupiter and the comet; the unphysical regions are in dark grey. Dashed lines indicate the values $T = 2$ and 3 (see Fig. 14.11); the thin dashed line labelled $q = 2.5$ shows T for a 2.5 AU perihelion comet in the ecliptic (objects above and to the left of this curve are very difficult to detect because they never get close to the Sun). Adapted from H. Levison, Comet taxonomy, in: *Completing the Inventory of the Solar System*, eds. T.W. Rettig and J.M. Hahn (Astronomical Society of the Pacific), p. 173 (1996).

Fig. 7.— Populations of minor bodies in space $(1/a, T)$. Bertotti et al. 2003.

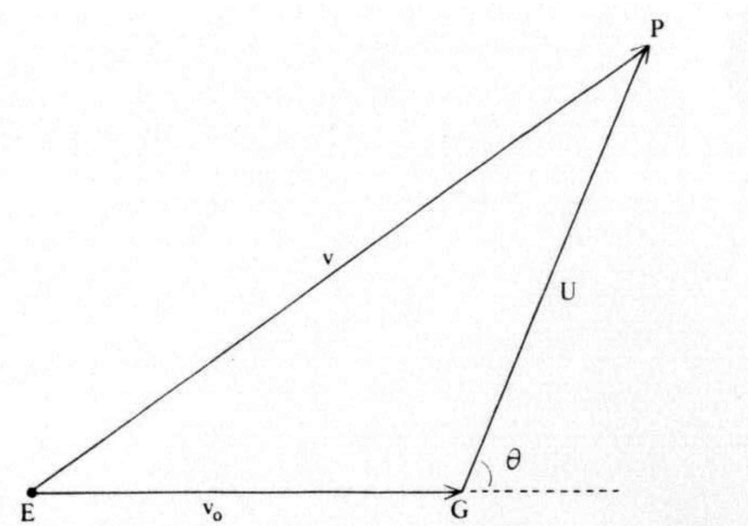

Fig. 11. Heliocentric velocity v as vector sum of U and v_0.

Fig. 8.— Öpik 1976.

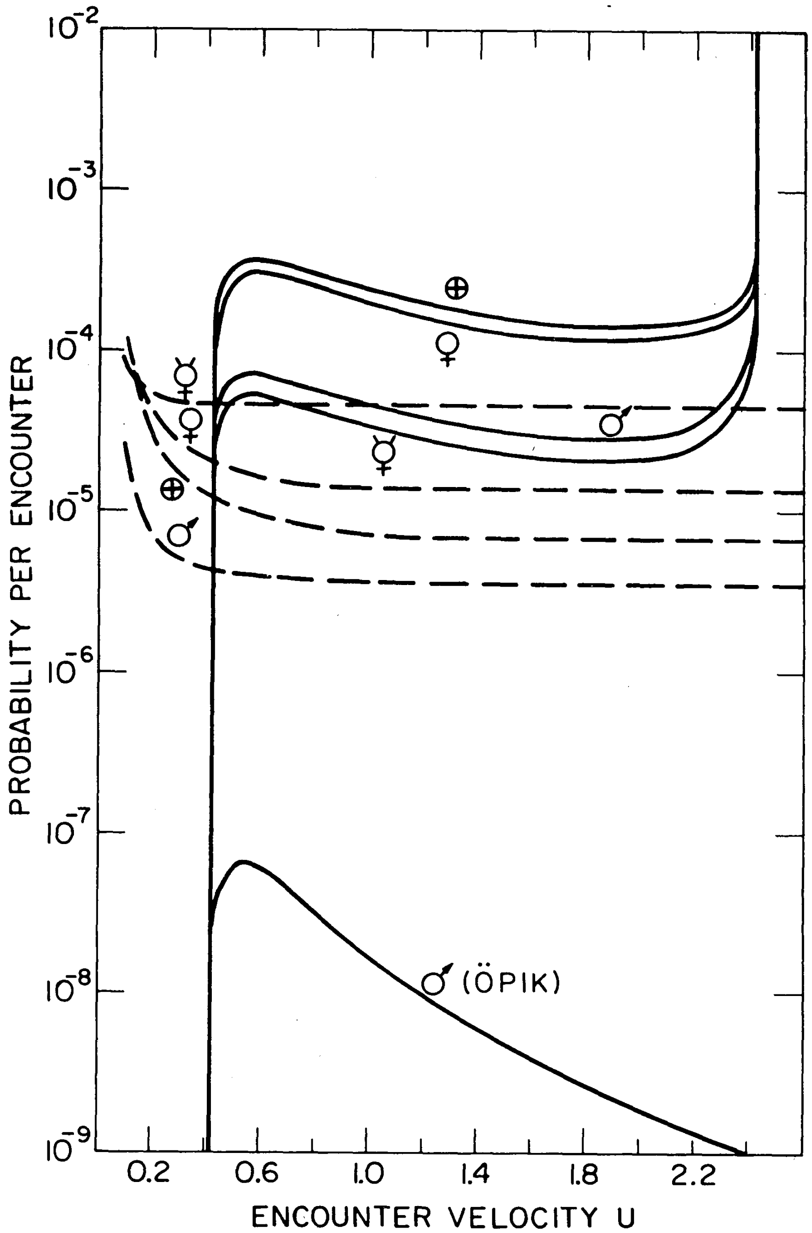

FIG. 5. Probabilities of collision (dashed lines) and ejection (solid lines) for terrestrial planets, from Eqs. 19 and 29, and for Mars from Eq. 42.

Fig. 9.— Collision versus ejection for terrestrial planets. Weidenschilling 1975.

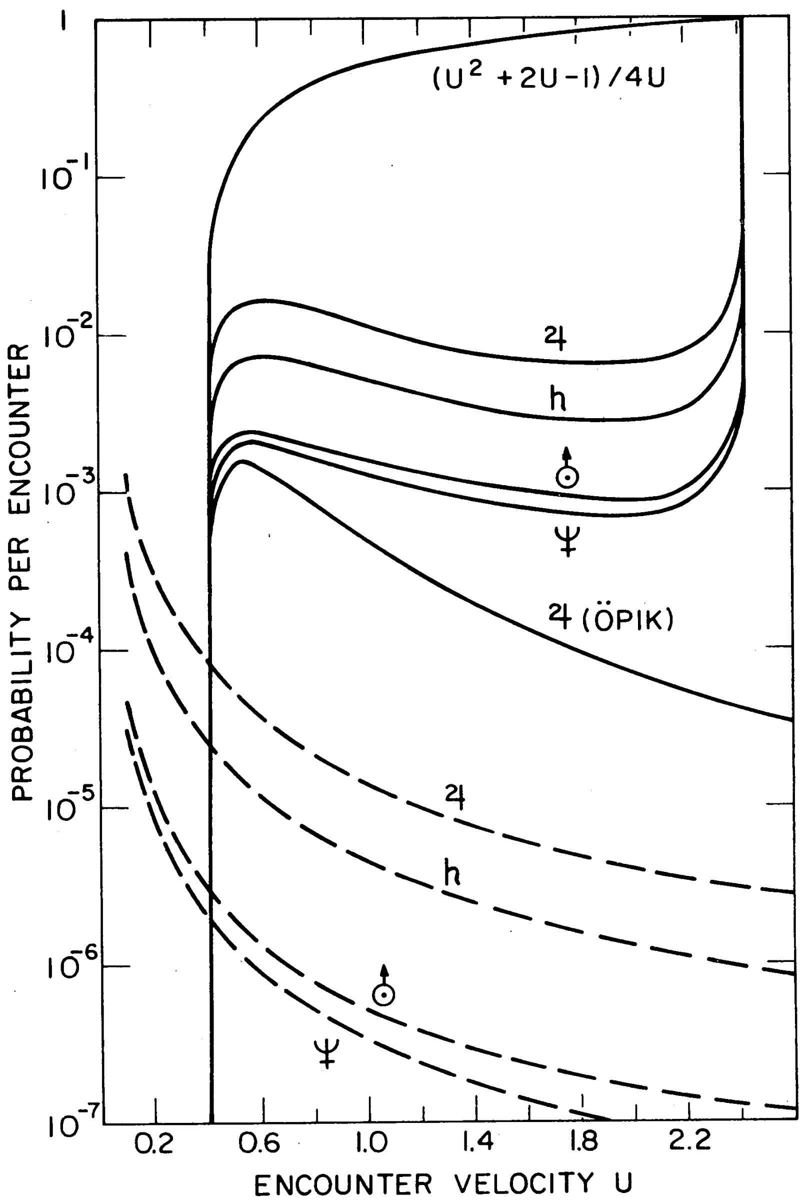

FIG. 4. Probabilities of collision (dashed lines) and ejections (solid lines) for Jovian planets, from Eqs. 19 and 29. Also shown are fractional solid angle of the escape cone and Öpik's ejection probability for Jupiter from Eq. 42.

Fig. 10.— Collision versus ejection for jovian planets. Weidenschilling 1975.

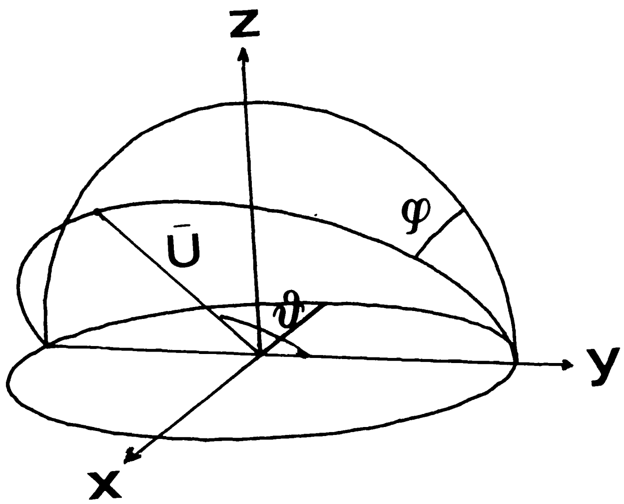

Fig. 1. The frame of reference of the vector **U**. The origin is at the planet's centre, the *y*-axis is oriented in the direction of motion of the planet, the *x*-axis is in the opposite direction with respect to the sun, the *z*-axis is parallel to the planet's angular momentum vector. The direction of **U** is provided by the two angles θ and φ.

Fig. 11.— Orientation of U in the rotating frame. Carusi et al. 1990.

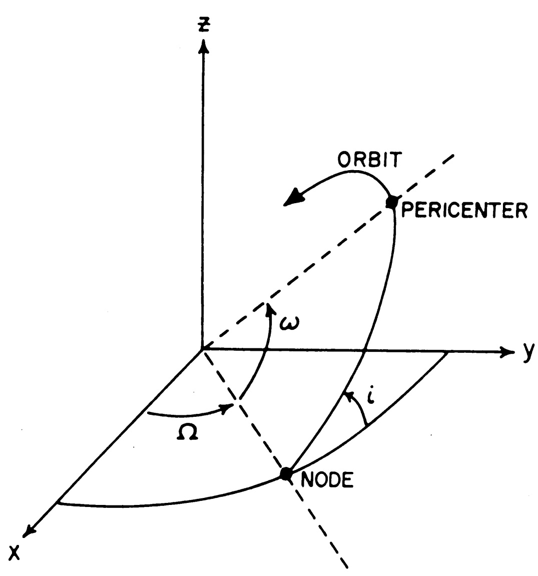

Fig. 12.— Angular orbital elements referred to an inertial system.

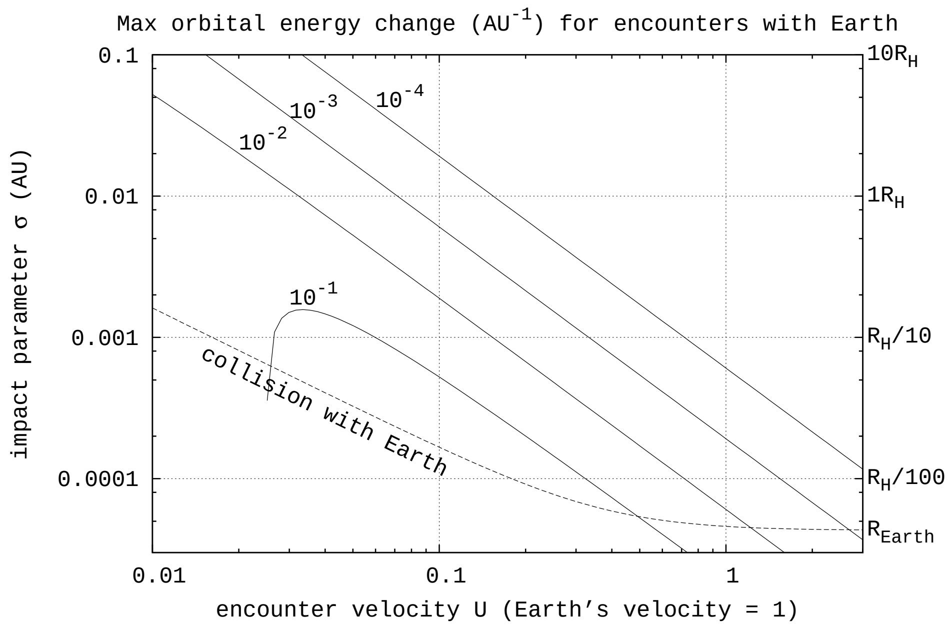

Fig. 13.— Maximum possible changes in orbital energy $x = 1/a$ for encounters with Earth.

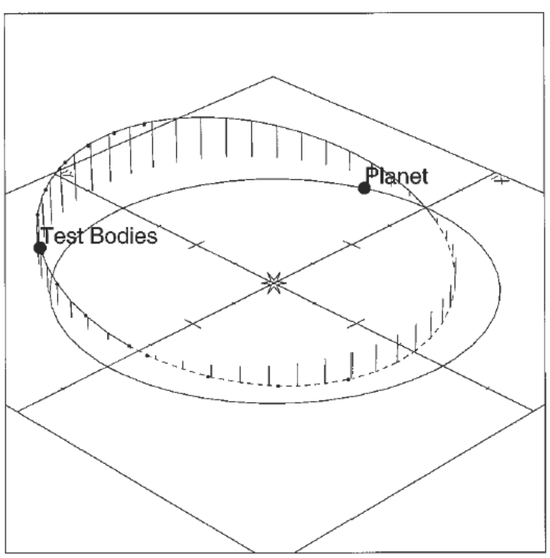

Fig. 14.— A collision orbit. Bottke et al. 1997.

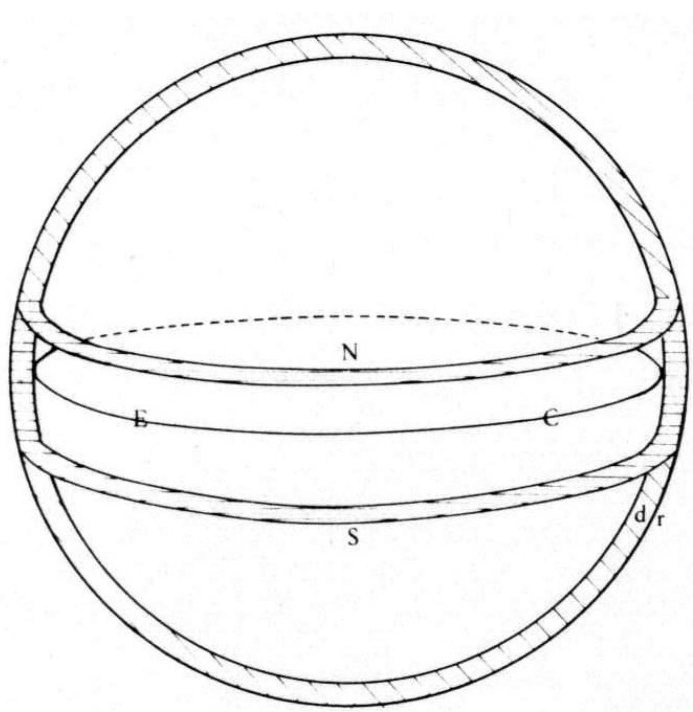

Fig. 5. The crossing volume of presence.

Fig. 15.— Crossing region. Öpik 1976.

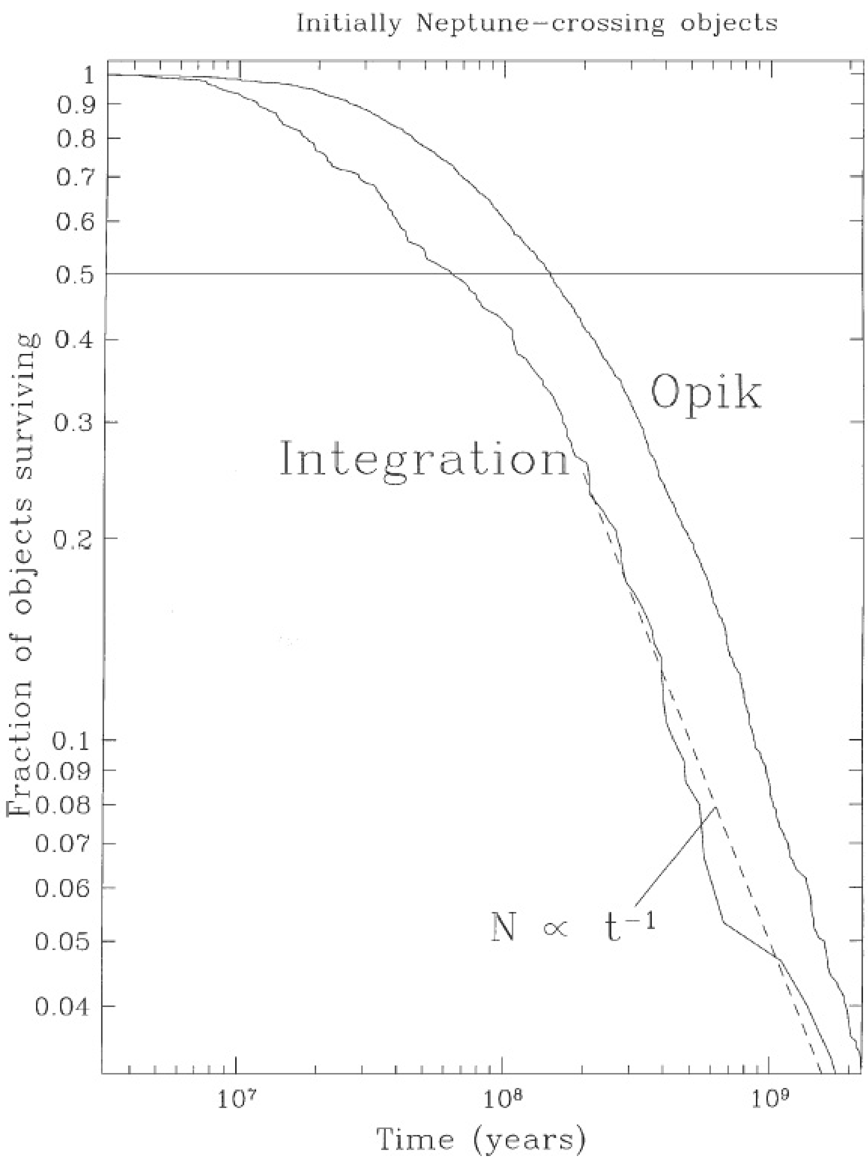

Fig. 16.— Fraction of survivors according to Öpik modeling and numerical integrations. Dones et al. 1999.

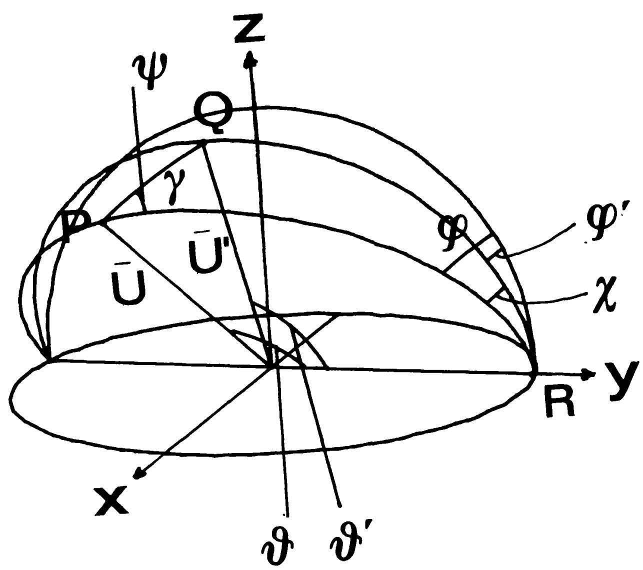

Fig. 2. After the encounter the vector **U** is rotated by an angle γ in the direction given by ψ. This last is the angle (counterclockwise) from the meridian RP, containing the velocity vector. After rotation, the direction of U is given by the angles φ' and θ'.

Fig. 17.— New and old encounter velocity U. Carusi et al. 1990.

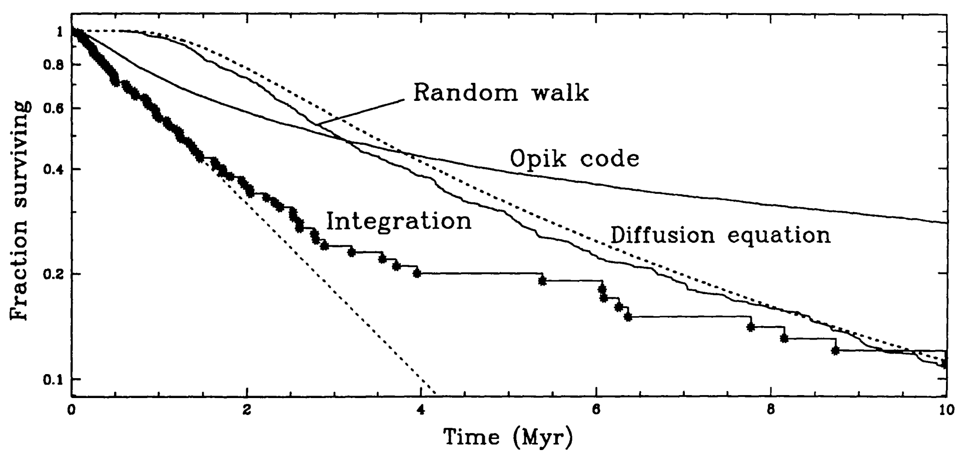

Figure 2. (a) Number of surviving objects as a function of time from our integrations. (b) Number of surviving Chirons vs. time from (a), plus the same quantity computed by an Öpik calculation and the diffusion approximation. For comparison with the diffusion approximation, we also have computed a symmetric random walk in energy with a fixed step size equal to $\sqrt{\langle\sigma_x^2\rangle}$. In the integrations, exponential decay (shown by the straight line) is a good approximation at early times ($t < t_{\rm med}$).

Fig. 18.— Fraction of survivors according to different models. Dones et al. 1996.

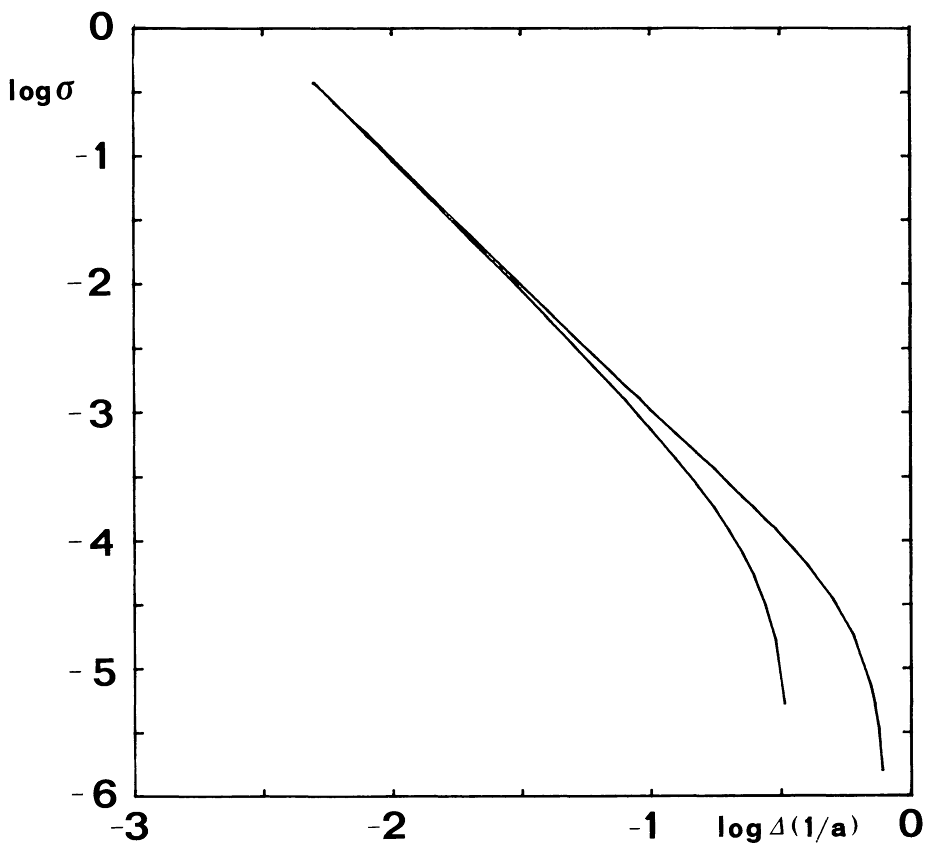

Fig. 9. Log–log plot of the cross section σ of fictitious comets scattered by Jupiter versus the corresponding $(1/a)$ perturbation per unit mass; upper curve: negative variations; lower curve: positive variations. This figure should be compared with Everhart (1969) Figure 1. Arbitrary units.

Fig. 19.— Asymmetric tails in the distribution of Δx. Carusi et al. 1990.

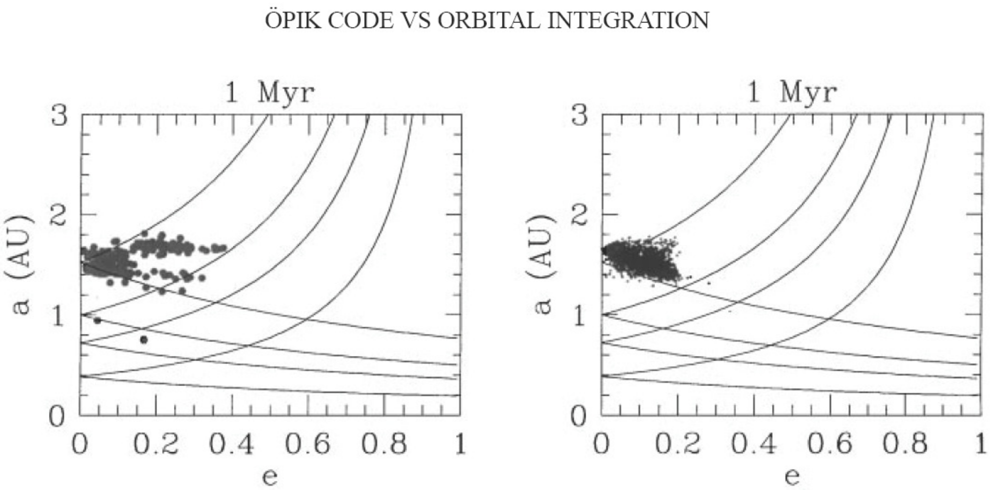

Fig. 20.— Öpik method cannot take account for resonances. Dones et al. 1999.

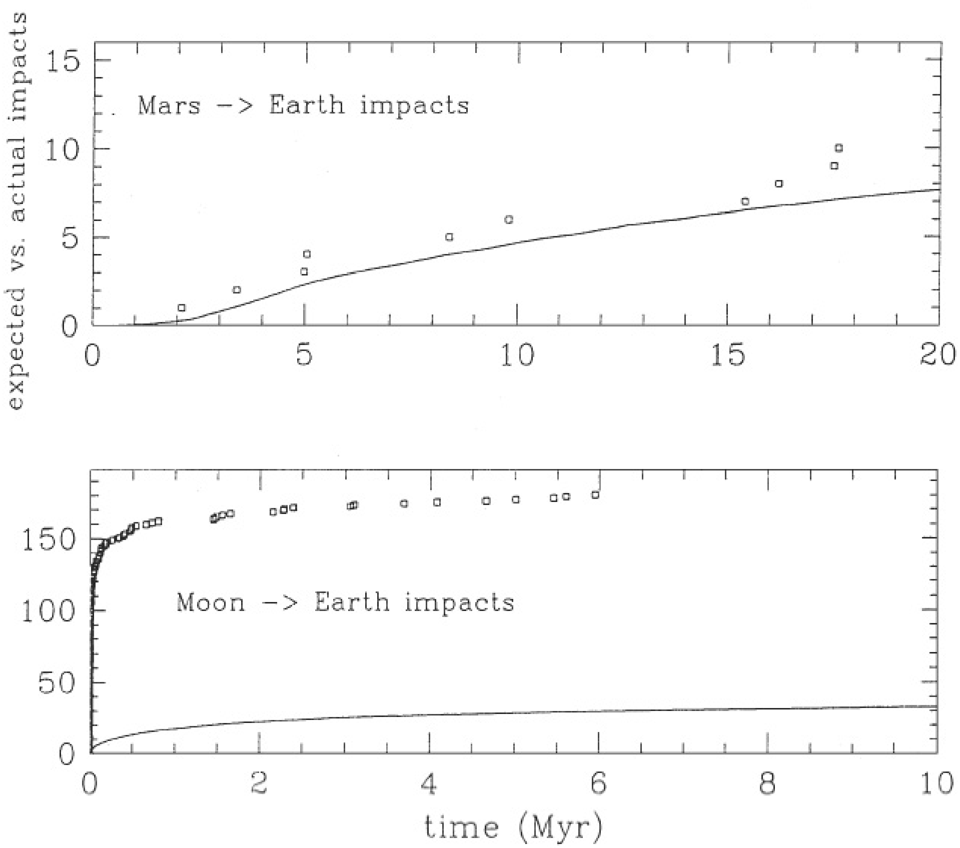

Fig. 21.— Öpik method fails when U is near zero. Dones et al. 1999.