Turbulent Advection of Reacting Substances

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Abstract

The turbulent advection of a reacting substance is considered. The existence of two different states, one smooth and the other filamental, is discussed. The transition between these states is characterized in terms of the competition of the stretching and folding mechanism of the turbulent field and the tendency of the scalar field to relax to the non-homogeneous steady state.

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Mixing in fluids has attracted much interest in recent years and particular considerable progress has been made concerning chaotic advection [1–3]. A remarkable fact recently pointed out is the existence of a smooth-filamental transition of active tracers stirred by chaotic advection [4]. Here, we are interested in the existence of a similar transition when the advection is produced by a turbulent field. As a relevant example of this kind of situation we consider the model proposed by Abraham concerning plankton dynamics [5].

We consider here the turbulent advection of a reacting scalar field $C(\vec{r},t)$. The scalar field is assumed to be passive, i.e. it does not affect the turbulent velocity field and its dynamics to be stable, that is, in the absence of the advecting field the system relaxes to a non-homogeneous steady state $C_0(\vec{r})$. The reaction-advection equation for the scalar field $C(\vec{r},t)$ reads

$$\frac{\partial C}{\partial t} + \vec{v}(\vec{r},t) \cdot \vec{\nabla} C = \alpha (C_0(\vec{r}) - C), \quad (1)$$

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where $\vec{v}(\vec{r},t)$ denotes an incompressible ($\nabla \cdot \vec{v} = 0$) two-dimensional turbulent field and α is the relaxational ratio to the non-homogeneous steady state representing an inhomogeneous forcing at large scales which may give account of upwelling, solar radiation, etc. In this presentation we have chosen a simple form for $C_0(\vec{r})$ given $C_0(x,y) = 1 + A\sin(2\pi x n_l/L)\sin(2\pi y n_l/L)$ where A and n_l are constants and $x, y \in (0, L)$. In our model we do not include a molecular diffusion term. The inclusion of a small molecular diffusivity in the equation of motion for the scalar field tends to smooth out inhomogeneities at scales smaller than the diffusion length scale.

The turbulent field is generated in a kinematic approach following a stochastic method based on a generalized Langevin equation [6]. Following this method, the physical parameters of the velocity field, the intensity u_0^2, the integral length l_0, the correlation time t_0, and the energy spectrum $E(k)$ can be arbitrarily chosen. In this presentation we consider a geophysical two-dimensional spectrum given by

$$E(k) \propto k^3 \left[1 + \lambda^2 k^2\right]^{-3}. \tag{2}$$

This spectrum follows the widely accepted law of potential decay of the form $\sim k^{-3}$ within a range of scales over which the enstrophy cascades towards higher wavenumbers [7,8]. The parameter λ, related to the maximum value of $E(k)$ introduces a typical length scale in the spectrum.

Concerning the numerical integration of the reaction-advection equation (1) we use the two-step Lax-Wendroff method with a small time step to ensure the stability. The dynamics of the turbulent field is simulated in Fourier space with periodic boundary conditions in both dimensions. A detailed presentation of the algorithm implemented to simulate turbulent flows can be found in [6].

Let us calculate the typical growth rate of the distance between two initially very close particles in the turbulent flow. We know from previous studies that the separation dynamics has three stages. For very short times the particles are moving in approximate straight lines, at intermediate times enhanced dispersion following Richardson's law applies, and, finally, for large times the particles become independent and diffusion prevails [10]. For short times where the separation is small compared with the characteristic length, l_0, the separation $\delta\hat{\vec{r}}$ is given by $|\delta\hat{\vec{r}}(t)| \sim e^{st}|\delta\hat{\vec{r}}(0)|$ where $s(\vec{r},t)$ is the local stretching rate. Actually, at a given point there are expanding and contracting directions, and if the initial conditions are along the contracting (expanding) direction the particles will get closer (farther) following $\exp(-st)$ ($\exp(st)$), so that s defines a sort of local Lyapunov exponent. The stretching rate s can be related to the physical parameters of the velocity field. After averaging it can be shown that $s \propto u_0/l_0$ where the proportionality constant depends on the details of the spectrum employed [11].

Following Refs. [4,9] we adopt a Lagrangian perspective to describe the temporal evolution of the scalar field by means of ordinary differential equations. In such a perspective the time dependent position of a marked fluid particle is denoted by $\hat{\vec{r}}(t)$. The Lagrangian version of the equation (1) gives the dynamics of the scalar fields within each parcel of fluid. In particular, the equation for the carrying capacity results

$$\frac{d\hat{C}}{dt} = \alpha(C_0(\hat{\vec{r}}) - \hat{C}) \tag{3}$$

To investigate the spatial structure of the scalar field we calculate the difference $\delta C = C(\vec{r} + \delta\vec{r}, t) - C(\vec{r}, t)$. Expanding Eq. (3) in the vicinity of a reference orbit and integrating [4] we obtain

$$\delta C(t) = \delta C(0)e^{-\alpha t} + \int_0^t \alpha \nabla_r C \cdot \delta\vec{r}(t')e^{\alpha(t'-t)}dt'. \tag{4}$$

This equation can be integrated knowing the separation of the particles at previous times, $\delta\hat{\vec{r}}(t')$ $(0 < t' < t)$. In consequence, using Eq. (4) one can obtain the directional derivative along the direction $\vec{n} = \delta\vec{r}/|\delta\vec{r}|$

$$\frac{\delta C(t)}{|\delta\vec{r}(t)|} = \delta C(0)e^{(s-\alpha)t} + \int_0^t \alpha \nabla_r C \cdot \vec{n}\, e^{(s-\alpha)(t-t')}dt' \tag{5}$$

If $s \lesssim \alpha$ the derivative remains finite and we obtain a smooth pattern. On the contrary, when $s > \alpha$ the derivative diverges in all the directions except in the contracting direction, then, we obtain a filamental case.

In order to numerically verify the above reasoning let us consider two examples corresponding respectively to the smooth and filamental cases. Starting from a random initial condition the scalar field evolutionates according to Eq. (1). In figure 1 we observe the spatio-temporal patterns representing the scalar concentration in the stationary state ($t = 40$) for two values of α considered. In the first case ($s < \alpha$, left) we observe that the distribution is smooth. In the second case, ($s > \alpha$, right) the distribution is filamental.

In conclusion, we have studied the patterns formed by reacting scalar fields under the influence of turbulent flows. We have analyzed a smooth-filamental transition in terms of the competition between stretching and folding mechanism of the velocity field and relaxation to the equilibrium of the scalar field. The relevant parameters of the transition are the intensity and the characteristic length of the velocity field and the relaxational ratio α. The role of the molecular diffusivity is to smooth out small scales.

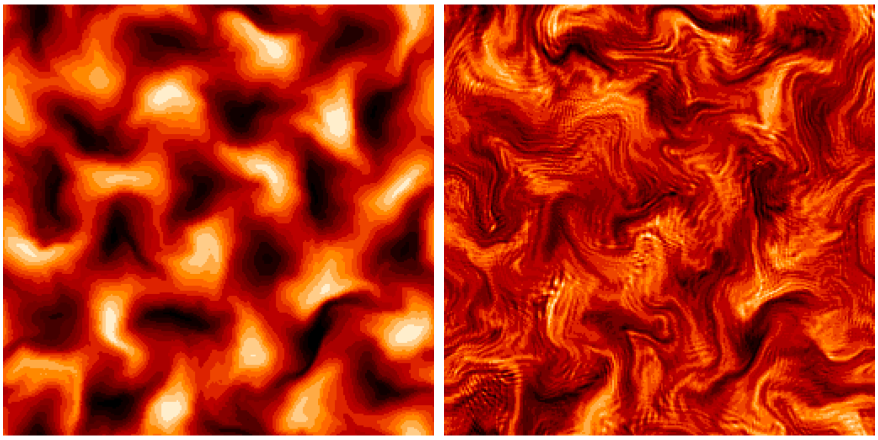

Fig. 1. Scalar concentration, for $s = 0.34$ and $\alpha = 0.25$ (left) and $\alpha = 0.025$ (right). The parameter of the velocity field are $u_0^2 = 4.0$, $t_0 = 5.6$, and $l_0 = 8.0$, $N = 256$, and $\Delta = 0.5$.

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