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Synchronization in an array of globally coupled maps with delayed interactions

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Abstract

We study synchronization of a one-dimensional array of coupled logistic maps in the regime where the individual maps, without coupling, evolve in a periodic orbit. We investigate the effect of a delay in the coupling that takes into account the finite velocity of propagation of interactions. Two qualitatively different synchronization regimes are found, depending on the value of the coupling strength. For weak coupling the array divides into clusters, and the behavior of the individual elements within each cluster depends on the delay times. For strong enough coupling, the array synchronizes into a single cluster. The evolution of the elements is periodic and their relative phases depend on the delay times.

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Ensembles of coupled dynamical elements provide the basis of a wide class of mathematical models for natural complex phenomena, ranging from nonequilibrium macroscopic physical processes to biological evolution and ecology dynamics [1,2]. Over the last two decades, in particular, globally coupled systems have attracted particular attention. In globally coupled systems, the interaction—which is usually introduced as an attractive mutual action—does not depend on the distance between dynamical elements,

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and is typically defined through a set of quantities that correspond to global averages over the ensemble. These models describe systems where the interaction ranges are of the order of the system size, as it happens to be in many instances of biological origin. The prototypical manifestation of collective behavior in globally coupled systems is synchronization, where the dynamical elements converge to a single trajectory in phase space [3–5]. This behavior qualitatively reproduces synchronization phenomena in real systems such as in neural networks and biological populations. In globally coupled ensembles, synchronization occurs above a certain interaction strength. The synchronization regime is usually preceded by a range where the ensemble is partially synchronized, forming clusters of mutually synchronized elements [5–7].

In many physical and biological processes, time delays play an essential dynamical role. For instance, in spite of the fact that in biological populations interaction lengths may be comparable to the system size, interaction carriers—such as sound or odor—can be relatively slow, directly affecting the collective behavior of the population. It is, therefore, relevant to study globally coupled systems with time delays, as already done in some previous work [8–10]. In the present paper, we study a model where time delays in a globally coupled ensemble are directly associated with the spatial distribution of the ensemble [11]. We consider a one-dimensional array of N maps, coupled as

$$x_i(t+1) = (1-\varepsilon) f[x_i(t)] + \frac{\varepsilon}{N} \sum_{j=1}^{N} f[x_j(t-\tau_{ij})] \; , \qquad (1)$$

where $f(x) = ax(1-x)$ is the logistic map, ε is the strength of the coupling, and τ_{ij} is a time delay, proportional to the distance between the ith and jth maps. Assuming that the boundaries in the array are free, we take $\tau_{ij} = k|i-j|$. Here k is the inverse of the velocity of the interaction signal traveling along the array.

In this preliminary report, we limit ourselves to the case where the parameter a is such that the individual logistic maps, without coupling, evolve in a limit cycle of period $P = 2$, i.e., $3 < a < 1+\sqrt{6}$. In such case, there are two exact solutions of Eq. (1) which exist for all values of ε. These solutions are characterized by the fact that, for all pairs i, j, the signal received by map i at each time corresponds to a *delayed* state of map j that coincides with the *present* state of map i:

$$x_j(t-\tau_{ij}) = x_i(t) \; . \qquad (2)$$

Thus, each map "perceives" the ensemble as being fully synchronized. In both solutions, moreover, each map evolves along the limit cycle of period 2 of the uncoupled dynamics.

The first of the two solutions corresponds to an actually synchronized state, $x_1(t) = x_2(t) = \cdots = x_N(t)$ for all t. This *in-phase* solution exists for k even, where τ_{ij} is even for all i, j and, thus, delays are irrelevant to the dynamics. For k odd, on the other hand, we find an *anti-phase* exact solution, where the ensemble divides into two clusters. The first cluster includes the elements at even positions in the array, which are mutually synchronized. The second cluster contains the maps at the odd sites. They are also synchronized, but evolve along the limit cycle in counter-phase with respect to the element of the first cluster. The successive states of the two clusters can be illustrated as follows:

$$\text{first cluster (even positions)} : x_A x_B x_A x_B \cdots ,$$

$$\text{second cluster (odd positions)} : x_B x_A x_B x_A \cdots .$$

Here, x_A and x_B are the states of the limit cycle of period 2. We refer to these two solutions—which, as shown below, are frequently found in numerical realizations of our system—as the *synchronized* states of the ensemble.

To investigate the stability of the synchronized solutions, we have performed numerical realizations starting from slightly perturbed in-phase and anti-phase states. The integration of the delay equations (1) requires to specify the evolution of $x_i(t)$ along the interval $\max(\tau_{ij}) \leqslant t \leqslant 0$. We have defined this initial evolution by letting the array evolve without coupling from given intial conditions. Our realizations were performed for an array of $N=100$ maps with $a=3.2$, for several values of k and ε. We have found that, independently of the value of the coupling constant ε, the in-phase and anti-phase solutions are asymptotically approached for k even and odd, respectively. Thus, both synchronized solutions are stable for the respective values of k.

On the other hand, random initial conditions—where the initial state of each element is chosen at random in $(0,1)$—are typically not attracted by the synchronized in-phase or anti-phase solutions when coupling is weak ($\varepsilon < 0.1$). The typical asymptotic solutions for these initial conditions are the clustered states described in the following. For small ε and given k, thus, a synchronized solution coexist with the clustered states and the system is multistable. As ε grows, however, the basin of attraction of the synchronized solution increases in size and the fraction of initial conditions that converge to a clustered state decreases accordingly.

In the clustered states, for any k and ε, each map is found to perform a cycle of period 2. The detailed dynamical nature of the clustered states depends however on whether k is even or odd. For k even, the elements of the array divide into two clusters of sizes N_1 and N_2 ($N_1+N_2=N$, typically with $N_1 \approx N_2 \approx N/2$). While the elements of one cluster visit the points x'_A and x'_B the elements of the other cluster visit the points x''_A and x''_B. Moreover, while the elements of the first cluster are in state x'_B, the elements of the other cluster are in x''_A, and vice versa. Taking into account that the dynamics of the individual maps is periodic of period 2, and that τ_{ij} is even for all i and j, the values x'_A, x'_B, x''_A, and x''_B for a given partition (N_1, N_2) can be calculated from the equations

$$\begin{aligned}
x'_A &= (1-\varepsilon) f(x'_B) + \frac{\varepsilon}{N}[N_1 f(x'_B) + N_2 f(x''_A)] , \\
x'_B &= (1-\varepsilon) f(x'_A) + \frac{\varepsilon}{N}[N_1 f(x'_A) + N_2 f(x''_B)] , \\
x''_A &= (1-\varepsilon) f(x''_B) + \frac{\varepsilon}{N}[N_1 f(x'_A) + N_2 f(x''_B)] , \\
x''_B &= (1-\varepsilon) f(x''_A) + \frac{\varepsilon}{N}[N_1 f(x'_B) + N_2 f(x''_A)] .
\end{aligned} \tag{3}$$

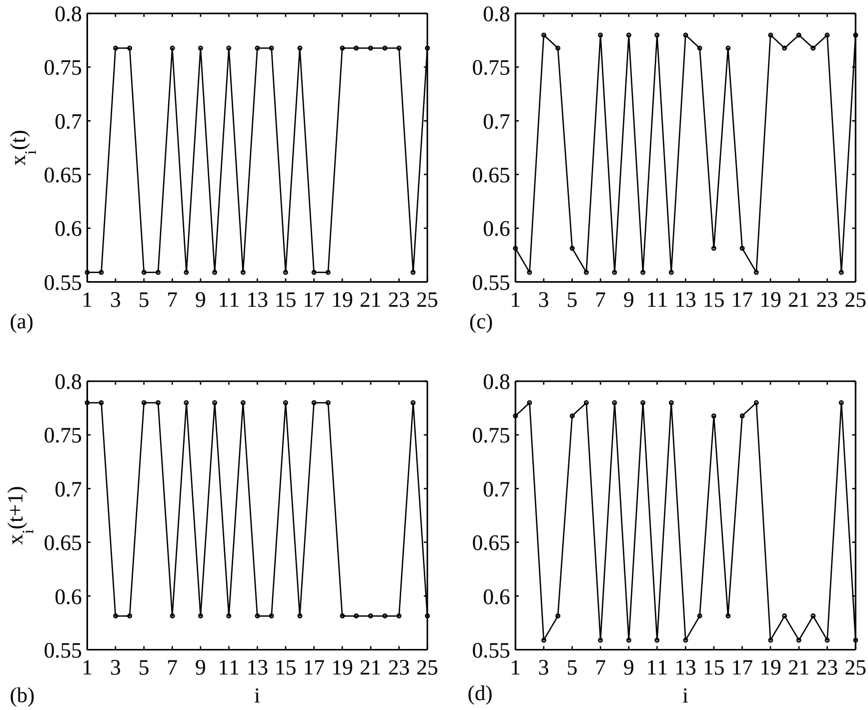

Fig. 1. Clustering behavior for weak coupling. We show two successive states of a linear array for ((a), (b)) $k=2$, ((c), (d)) $k=1$. The other parameters are $N=100$, $a=3.2$, and $\varepsilon=0.09$.

Since the spatial position of the elements of each cluster is arbitrary, each partition is associated, for large N, with a multitude of clustered states. Figs. 1(a) and (b) show two consecutive snapshots of a clustered state for the first 25 elements in the array, with $k=2$ and $\varepsilon=0.09$.

A somewhat more complicated behavior is found when k is odd. After a transient the elements of the array divides into four clusters of sizes $N_1 \cdots N_4$ ($N_1+\cdots+N_4=N$, typically with $N_1 \approx \cdots N_4 \approx N/4$). As before, within each cluster the individual dynamics is periodic of period 2. Now, the positions of the elements of a given cluster are all even or odd. It is observed that one of the clusters whose elements are at even positions and one of the clusters whose elements are at odd positions visit states x'_A and x'_B, in such a way that when the elements of the first cluster are in x'_A the elements of the other cluster are in x'_B, and vice versa. The remaining two clusters behave analogously between states x''_A and x''_B. To clarify this rather obscure picture, we give an explicit example of the successive states of the four clusters:

$$\text{cluster of size } N_1 \text{ (odd positions)} : x'_A x'_B x'_A x'_B \cdots ,$$

$$\text{cluster of size } N_2 \text{ (even positions)} : x''_A x''_B x''_A x''_B \cdots ,$$

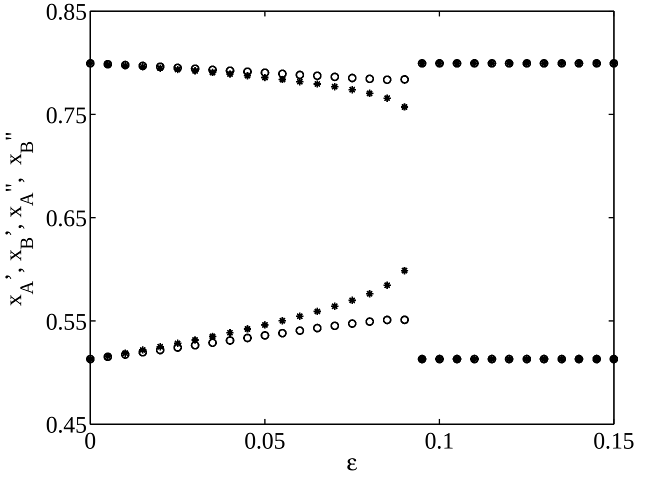

Fig. 2. Dependence of x'_A, x'_B (circles), x''_A, x''_B (stars) with the coupling strength. We chose the *same* random initial condition for all ε. Above a certain threshold, $\varepsilon \approx 0.09$, the data stand for the states of the in-phase solution. The parameters are $N = 100$, $a = 3.2$, and $k = 2$.

$$\text{cluster of size } N_3 \text{ (odd positions)} : x''_B x''_A x''_B x''_A \cdots ,$$

$$\text{cluster of size } N_4 \text{ (even positions)} : x'_B x'_A x'_B x'_A \cdots .$$

The states x'_A, x'_B, x''_A, and x''_B for a given partition $(N_1, \ldots, N_4)$ are given by the equations

$$x'_A = (1-\varepsilon) f(x'_B) + \frac{\varepsilon}{N} [M_1 f(x'_B) + M_2 f(x''_A)] ,$$

$$x'_B = (1-\varepsilon) f(x'_A) + \frac{\varepsilon}{N} [M_1 f(x'_A) + M_2 f(x''_B)] ,$$

$$x''_A = (1-\varepsilon) f(x''_B) + \frac{\varepsilon}{N} [M_1 f(x'_A) + M_2 f(x''_B)] ,$$

$$x''_B = (1-\varepsilon) f(x''_A) + \frac{\varepsilon}{N} [M_1 f(x'_B) + M_2 f(x''_A)] \tag{4}$$

with $M_1 = N_1 + N_4$ and $M_2 = N_2 + N_3$. Note that these equations are formally the same as Eqs. (3). The solutions will therefore be numerically identical. Again, clustered states for k odd conform a multitude of possible asymptotic solutions of our system. Figs. 1(c) and (d) display two consecutive states of the array for $k = 1$ and $\varepsilon = 0.09$.

Taking a *fixed* random initial condition—which in turn fixes the partition in clusters—the values x'_A, x'_B, x''_A and x''_B vary with ε as illustrated in Fig. 2, where we plot for $k = 2$ the values of $x_i(t)$ and $x_i(t+1)$ (with t large enough) for two elements from different clusters. Above a certain coupling intensity, $\varepsilon \approx 0.09$, the array synchronizes in-phase. This critical value is found to depend on the initial condition, as expected from the multistable nature of the system for small ε.

To summarize, we have studied the effect of delayed interactions in a linear chain of globally coupled logistic maps. We considered the case in which the maps, without coupling, evolve in a limit cycle of period $P = 2^n$. We found global synchronization as well as clustering behavior, and in both cases the relative evolution of the maps in the array depend on the delay times. When the delay times τ_{ij} are all even (i.e., when $\tau_{ij} = k|i-j|$ and k is even) in the globally synchronized state the present state of all the maps coincide. When k is odd (and therefore, the delay times are either even or odd, depending on $|i-j|$), in the globally synchronized state there is a constant difference between the states of the maps. In both cases, the state of map j at time $t - \tau_{ij}$ coincides with the present state of map i (for all i, j and t). The study of synchronized states for arbitrary values of P and in the chaotic regime of logistic maps is the aim of future work.

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