THE CHANDRASEKHAR LIMIT FOR WHITE DWARFS

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Abstract. The aim of this work is to solve the set of differential equations that rules the behavior of white dwarfs and to find the limit mass for such peculiar stars. To solve this set of two non-linear differential equations we used the fourth order *Runge-Kutta Method* running under Matlab 6. In order to solve properly this equations we wrote them in terms of the dimensionless variables $\mu = \rho/\rho_c$, $\xi = r/R_{\odot}$ and $\zeta = m/M_{\odot}$ where ρ_c, $R_{\odot}$ and $M_{\odot}$ stands as the Sun's central density, radius and mass respectively. In this report we are going to find how the density and mass behaves as functions of the radial coordinate for a given value of its central density ρ_c, the radius of such stars, the relationship between their Radius and total Mass and as a final result, the limit mass known as the *Chandrasekhar Mass*. Founded by Subrahmanyan Chandrasekhar in the early 30's this limit mass of about $1.4M_{\odot}$ became fundamental for the understanding of the final stages of massive stars and gave a glimpse over new fields of astrophysics.

1. Introduction

Stars with masses that are not to large compared with the mass of the Sun end their lives as white dwarfs when they run out of their thermonuclear fuel. The core of such stars consist mainly of carbon and oxygen due to the outcomes of the *triple-alpha* process and by *radiative capture* of ^4He and by ^{12}C respectively. The pressure needed to prevent the gravitational collapse can be assumed to come from the pressure of the degenerate electron gas in the interior of the star. The white dwarfs are the last stage of stars with masses close to the mass of the sun, e.g., Sirius B and Eri B. This compact objects are the remanent core of red giant stars when they get rid of their external layers in a stage known as planetary nebulae. Containing central densities of the order 10^8-10^{12} kgm^{-3} this stars present a radius of the order $10^{-2}R_{\odot}$. We start by considering the basic set of equations needed to make a simple model of a white dwarf. This simplified model assumes a core consisted of completely ionized ^{12}C, a spherical non-rotating star and neglects effects due to the presence of magnetic fields. As before we also assume that the pressure is due to the degenerate electrons. If the a star is at Hydrostatic equilibrium the gravitational force on each mass element should be equal to the pressure gradient again at each mass element. The modulus of the gravitational force acting at distance r per volume of mass is given by:

$$F_{grav} = -\frac{Gm\rho}{r^2} \tag{1.1}$$

Where G, m and ρ stands as the gravitational constant, mass and density respectively.

As said before, at hydrostatic equilibrium conditions this gravitational force should be equal to the pressure gradient. Thus using (1.1) we get the equation for *Hydrostatic equilibrium*

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2} \tag{1.2}$$

The mass within a sphere of radius r is given by:

$$m(r) = 4\pi \int_0^r \rho(r')r'^2 dr' \tag{1.3}$$

Taking the derivative of this expression we get *the mass continuity equation*:

$$\frac{dm}{dr} = 4\pi r^2 \rho(r) \tag{1.4}$$

The pressure due to the degenerate electrons would depend if one consider a electron gas of *Non Relativistic* or *Ultra Relativistic* particles. Therefore we need a model of a white dwarf that consider both extreme cases. To do so we express the kinematics of the degenerate electron gas as:

$$\varepsilon_p^2 = m_e^2 c^4 + p^2 c^2$$

then expressing the pressure in terms os the density of states $g(p)$ as $V/h^3 4\pi p^2$ we get

$$P = \frac{1}{3V} \int_0^{P_F} \frac{p^2 c^2 g(p)}{\varepsilon_p} dp \tag{1.5}$$

one obtains

$$\begin{aligned} P &= \frac{8\pi}{3h^3} \int_0^{P_F} p^4 c^2 (m_e^2 c^4 + p^2 c^2)^{-1/2} dp \\ &= \frac{8\pi}{3h^3} \int_0^{P_F} p^4 c^2 [1 + (\frac{p}{m_e c})^2]^{1/2} m_e c^2 dp \end{aligned} \tag{1.6}$$

applying a variable change: $x = \frac{p}{m_e c}$, $dx = \frac{dp}{m_e c}$ yields

$$P = \frac{8\pi m_e^4 c^5}{3h^3} \int_0^{x_F} \frac{x^4}{(1+x^2)^{1/2}} dx \tag{1.7}$$

Some integration and a bit of tidying-up, leads to the following expression for the pressure of a degenerate electron gas

$$P = K_{UR} n_e^{4/3} I(x_f) \tag{1.8}$$

where

$$I(x) = \frac{3}{2x^4}[x(1+x^2)^{1/2}(2/3x^2 - 1) + ln[x + (1+x^2)^{1/2}]]$$

Here

$$K_{UR} = \frac{hc}{4}(\frac{3}{8\pi})^{1/3}$$

where x_F is the dimensionless *Fermi momentum*:

$$x_F = \frac{P_F}{m_e c} = (\frac{3n_e}{8\pi})^{1/3}\frac{h}{m_e c}$$

and $n_e$is the number of electrons per unit volume. This number is related to the density using Y_e as the number of electrons per nucleon

$$n_e = \frac{Y_e \rho}{m_H}$$

Finally we express the *Fermi momentum* as a function of the density

$$x_F = (\frac{3Y_e}{8\pi m_H})^{1/3}\frac{h}{m_e c}\rho^{1/3} \tag{1.9}$$

This pressure is now expressed in terms of the dimensionless *Fermi momentum* x_F, which depends on the density as shown. When considering higher densities $x_F \gg 1$ then $I(x_F) \to 1$, thus $P \to K_{UR} n_e^{4/3}$, at low densities, $x_F \ll 1$ and then $I(x_F) \to 4/5 x_F$ thus $P \to K_{NR} n_e^{4/3}$

Now we are ready to find how ρ should change as a function of the radial coordinate and x_F. We start by taking the radial derivative of the equation (1.8)

$$\frac{dP}{dr} = \frac{d}{dr} K_{UR} n_e^{4/3} I(x_F) \tag{1.10}$$

$$= \frac{d}{dr} K_{UR} \frac{Y_e}{m_H}^{4/3} \rho(r)^{4/3} I(x)$$

$$= K_{UR} \frac{Y_e}{m_H}^{4/3} (4/3 \rho^{1/3} \frac{d\rho}{dr} I(x) + \rho^{4/3} \frac{dI}{dx} |_{x_F} \frac{dx_F}{dr})$$

Where

$$\frac{dI}{dx} = \frac{4x^2+6}{2x^4(1+x^2)^{1/2}} - \frac{6ln[(1+x^2)^{1/2}+x]}{x^5}$$

and

$$\frac{dx_F}{dr} = 1/3 K_F \rho^{-2/3} \frac{d\rho}{dr}$$

From here we assume that $Y_e = 0.5$, the star consists of completely ionized ^{12}C. Writing again this pressure gradient we find the following equation:

$$\frac{dP}{dr} = K_{UR}(\frac{0.5}{m_H})^{4/3}(\frac{4\rho^{1/3}}{3}\frac{d\rho}{dr}I(x) + \rho^{2/3}\frac{K_F}{3}\frac{d\rho}{dr}\frac{dI}{dx}) \tag{1.11}$$

Where

$$K_F = (\frac{3*0.5}{8\pi m_H})^{1/3}\frac{h}{m_e c}$$

Now, being *EXTRA CAREFUL*, one can rewrite this equation by just trying to find some *hidden* x_F by using the density and K_F. After this considerations and having close to us the *Derive5* we can find the following friendly expression for the pressure gradient.

$$\frac{dP}{dr} = \frac{hc}{4}(\frac{3}{8\pi})^{1/3}(\frac{0.5}{m_H})^{4/3}\rho^{1/3}\frac{4x}{3(1+x^2)^{1/2}}\frac{d\rho}{dr} \tag{1.12}$$

Tidying up we get the pressure gradient for a degenerate electron gas.

$$\frac{dP}{dr} = \frac{0.5 m_e c^2}{3.m_H}\frac{x^2}{(1+x^2)^{1/2}}\frac{d\rho}{dr} \tag{1.13}$$

With this pressure gradient and the *mass continuity* equation we get the set of equations that will rule the behavior of a white dwarf.

$$\frac{d\rho}{dr} = \frac{-Gm(r)\rho(r)}{r^2\Upsilon(x)}\frac{m_H}{0.5c^2 m_e} \tag{1.14}$$

$$\frac{dm}{dr} = 4\pi r^2\rho(r) \tag{1.15}$$

Where

$$\Upsilon(x) = \frac{x^2}{3(1+x^2)^{1/2}}$$

The reader probably would be wondering why we bother to present Υ in this fashion. The reasons is because we still want the factor 0.5 coming from Y_e. So it could be easier to change this factor if we are asked to perform the same procedures for a star of different composition.

2. Obtaining the Dimensionless Equations $\frac{d\zeta}{d\xi}$, $\frac{d\mu}{d\xi}$

In order to solve the set of equations it is advisable to write them in terms of dimensionless quantities. Therefore let us introduce the following *dimensionless variables.*

$$\mu = \frac{\rho}{\rho_c} \tag{2.1}$$

$$\xi = \frac{r}{R_\odot} \tag{2.2}$$

$$\zeta = \frac{m}{M_\odot} \tag{2.3}$$

With this in mind let us rewrite the former set of equations by using our new set of variables μ, ξ and ζ. Lets start with the *mass continuity equation.* It follows directly from the definition of the *Fermi momentum* the following

$$x_F = K_F(\rho_c\mu)^{1/3} \tag{2.4}$$

then we can directly obtain from the former(2.1), our first dimensionless equation:

$$\frac{d\zeta}{d\xi} = \frac{4\pi\xi^2\mu R_\odot^3\rho_c}{M_\odot} \tag{2.5}$$

or it can be introduced as

$$\frac{d\zeta}{d\xi} = C_m \xi^2 \mu \tag{2.6}$$

where

$$C_m = \frac{4\pi R_\odot^3 \rho_c}{M_\odot}$$

For the remaining equation let us consider (1.14)

$$\frac{d\rho}{dr} = \frac{-Gm(r)\rho(r)}{r^2 \Upsilon(x)} \frac{m_H}{0.5c^2 m_e}$$

from there we start by considering the pressure gradient in (1.13)

$$\frac{dP}{dr} = \frac{0.5m_e c^2}{3.m_H} \frac{x^2}{(1+x^2)^2} \frac{d\rho}{dr}$$

in similar way featuring $\Upsilon(x)$

$$\frac{dP}{dr} = \frac{0.5m_e c^2}{m_H} \Upsilon(x) \frac{d\rho}{dr} \tag{2.7}$$

By a quick inspection we can pass from $\Upsilon(x)$ to $\Upsilon(\rho)$,

$$\Upsilon(\rho) = \frac{K_F^2 \rho^{2/3}}{3(1 + K_F^2 \rho^{2/3})^{1/2}} \tag{2.8}$$

From here we get

$$\Upsilon(\mu) = \frac{K_F^2 \rho_c^{2/3} \mu^{2/3}}{3(1 + K_F^2 \rho_c^{2/3} \mu^{2/3})^{1/2}} \tag{2.9}$$

Thus the pressure gradient goes from (1.13) to

$$\frac{dP}{d\xi} = \frac{0.5m_e c^2 \Upsilon(\mu)}{m_H} \frac{d\rho}{d\xi} \tag{2.10}$$

From the condition of *Hydrostatic equilibrium*, equation (1.2) we can easily get the same equilibrium condition in terms of μ and ξ.

$$\frac{dP}{d\xi} = \frac{-Gm(\xi)\mu\rho_c}{\xi^2 R_\odot} \tag{2.11}$$

Now we can rewrite the following

$$\frac{d\rho}{d\xi} = \frac{-Gm_H m(\xi)\mu\rho_c}{\xi^2 \Upsilon(\mu) 0.5m_e c^2 R_\odot} \tag{2.12}$$

From here it is easy to get the remaining dimensionless equation as

$$\frac{d\mu}{d\xi} = \frac{-Gm_H \mu M_\odot \zeta}{\xi^2 \Upsilon(\mu) 0.5m_e c^2 R_\odot} \tag{2.13}$$

As before this can be arranged as

$$\frac{d\mu}{d\xi} = C_\rho \frac{\zeta\mu}{\xi^2 \Upsilon(\mu)} \tag{2.14}$$

where

$$C_\rho = \frac{-Gm_H M_\odot}{0.5m_e c^2 R_\odot}$$

Finally we obtained the set of *dimensionless equations* with their respective dimensionless constants needed for an improved numerical computation as shown.

$$\frac{d\zeta}{d\xi} = C_m \xi^2 \mu \tag{2.15}$$

$$\frac{d\mu}{d\xi} = C_\rho \frac{\zeta\mu}{\xi^2 \Upsilon(\mu)} \tag{2.16}$$

$$C_m = \frac{4\pi R_\odot^3 \rho_c}{M_\odot} \tag{2.17}$$

$$C_\rho = \frac{-G m_H M_\odot}{0.5 m_e c^2 R_\odot} \tag{2.18}$$

Where

$$\Upsilon(\mu) = \frac{K_F^2 \rho_c^{2/3} \mu^{2/3}}{3(1 + K_F^2 \rho_c^{2/3} \mu^{2/3})^{1/2}}$$

Using our dimensionless quantities

$$\mu = \frac{\rho}{\rho_c}$$
$$\xi = \frac{r}{R_\odot}$$
$$\zeta = \frac{m}{M_\odot}$$

3. Solving the Equations

To solve the system of this two differential equations we wrote a code called *ecua.m* in Matlab. We implement the fourth order *Runge-Kutta Method*. The code is shown at the appendix for reference. To start the code one has to input a set of initial conditions, e.g, the central density and initial mass. As seen before, this quantities should be dimensionless, therefore one has to give at the beginning the central density in terms of the central density of the sun, ρ_c, and the value of the mass at $r = 0$. This initial mass it would be equal to zero. This selection of the *initial mass* implies that the *density gradient*, equation (1.16) or (2.16), will vanish at the origin of the radial coordinate. Thus the profile of the density as a function of radius should approach its maximum smoothly at the zero value of the radial coordinate. The radius of the star can be defined as the distance when the density reaches a zero value. I an similar way. the mass of the star can be defined as the value of the mass when the density reaches a zero value.

By using the *Runge-Kutta of fourth order Method* to solve de differential equations numerically we will not obtain a density profile that reaches zero but one that tends to. Thus we "cut" the density values at a certain point and after that we imposed that the rest of the values were zero. That non-zero value would correspond to the radius of the star. For practical comments see the program code.

We first test the program to see if its solves the equations correctly. As input values we gave the following ones:

μ	ζ	h(spacing iteration)
1e7	0	1e-5

Table 1:

Next we show some results obtained by ecua.m

μ	ξ	ζ
5e4	0.0133	0.54398
1e5	0.0115	0.67129
5e5	0.0079	0.9644
1e6	0.0066	1.07285
5e6	0.0044	1.2595
1e7	0.0039	1.3128
5e7	0.0022	1.3898
3e8	0.0013	1.4292
7e8	1.15e-3	1.43139
9.5e8	9e-4	1.4334

Table 2: Here we present some values obtained from the program corresponding to different central $\mu' s$

4. Conclusions

We see that the program works for h, the iteration step, of the order of 1e-5. By using a greater h the program misleads the results, giving smaller radii. This can be checked if we run the program for a know star. When assuming a stable white dwarf we suppose it is supported by the pressure of non-relativistic degenerate electrons. When such a start collapses degenerate electrons become relativistic as the density number of the electrons is larger that $(\frac{m_e c}{h})^3$. If we equate (1.8) for pressure of a degenerate electrons to the pressure needed to support a star of mass M we obtain,

$$M \simeq I(x)^{3/2} M_{ch} \tag{4.1}$$

This behavior is reflected in the third figure. The asymptotic curve shows that as the central density grows the mass reaches a critical value. This critical value obtained by ecua.m was $1.44 M_{\odot}$. This values in close agreement with theoretical ones for the *Chandrasekhar Mass*.

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